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Controlled L–theory

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We develop an epsilon-controlled algebraic L–theory, extending our earlier work on epsilon-controlled algebraic K–theory. The controlled L–theory is very close to being a generalized homology theory; we study analogues of the homology exact sequence of a pair, excision properties, and the Mayer–Vietoris exact sequence. As an application we give a controlled L–theory proof of the classic theorem of Novikov on the topological invariance of the rational Pontrjagin classes.

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1 Introduction

The purpose of this article is to develop a controlled algebraic L–theory, of the type first proposed by Quinn [8] in connection with the resolution of homology manifolds by topological manifolds.

We define and study the epsilon-controlled L–groups $L_n^{\delta,\epsilon}(X;p_X,R)$, extending to L–theory the controlled K–theory of Ranicki and Yamasaki [14]. When the control map p_X is a fibration and X is a compact ANR, these groups are stable in the sense that $L_n^{\delta,\epsilon}(X;p_X,R)$ depends only on p_X and R and not on δ or ϵ as long as δ is sufficiently small and $\epsilon \ll \delta$ (see Pedersen–Yamasaki [5]).

These are the candidates of the controlled surgery obstruction groups; in fact, such a controlled surgery theory has been established when the control map p_X is UV^1 (see Pedersen–Quinn–Ranicki [4] and Ferry [2]).

Although epsilon controlled L–groups do not produce a homology theory in general, they have the features of a generalized homology modulo controlled K–theory problems. In this article we study the controlled L–theory analogues of the homology exact sequence of a pair (Theorem 5.2), excision properties (Section 6), and the Mayer–Vietoris sequence (Theorem 7.3).

In certain cases when there are no controlled K–theoretic difficulties, we can actually show that controlled L–groups are generalized homology groups. This is discussed in Section 8.

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In the last two sections, we study locally-finite analogues and as an application give a controlled L–theory proof of the classic theorem of Novikov [3] on the topological invariance of the rational Pontrjagin classes.

2 Epsilon-controlled quadratic structures

In this section we study several operations concerning quadratic Poincaré complexes with geometric control. These will be used to define epsilon controlled L–groups in the next section.

In [14] we discussed various aspects of geometric modules and morphisms and geometric control on them, and studied K–theoretic properties of geometric (=free) and projective module chain complexes with geometric control. There we considered only $\mathbb{Z}$–coefficient geometric modules, but the material in Sections 1–7 remains valid if we use any ring R with unity as the coefficient. To incorporate the coefficient ring into the notation, the group $\widetilde{K}_0(X, p_X, n, \epsilon)$ defined using the coefficient ring R will be denoted $\widetilde{K}_0^{n,\epsilon}(X; p_X, R)$ in this article.

To deal with L–theory, we need to use duals. Fix the control map $p_X\colon\ M \to X$ from a space M to a metric space X and let R be a ring with involution (see Ranicki [10]). The *dual* G^* of a geometric R–module G is G itself. Recall that a geometric morphism is a linear combination of paths in M with coefficient in R. The *dual* f^* of a geometric morphism $f = \sum_\lambda a_\lambda \rho_\lambda$ is defined by $f^* = \sum_\lambda \overline{a}_\lambda \overline{\rho}_\lambda$, where $\overline{a}_\lambda \in R$ is the image of a by the involution of R and $\overline{\rho}_\lambda$ is the path obtained from ρ_λ by reversing the orientation. Note that if f has radius ϵ then so does its dual f^* and that $f \sim_\epsilon g$ implies $f^* \sim_\epsilon g^*$, by our convention. For a geometric R–module chain complex C, its n*-dual* C^{n-*} is defined using the formula in Ranicki [9].

For a subset S of a metric space X, S^ϵ will denote the closed ϵ neighborhood of S in X when $\epsilon \geq 0$. When $\epsilon < 0$, S^ϵ will denote the set $X - (X - S)^{-\epsilon}$.

Let C be a free R–module chain complex on $p_X\colon\ M \to X$. An *n–dimensional ϵ quadratic structure ψ on C* is a collection $\{\psi_s | s \geq 0\}$ of geometric morphisms

$$\psi_s\colon\ C^{n-r-s} = (C_{n-r-s})^* \to C_r \qquad (r \in \mathbb{Z})$$

of radius ϵ such that

$$(*) \quad d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) \sim_{3\epsilon} 0\colon\ C^{n-r-s-1} \to C_r,$$

for $s \geq 0$. An n–dimensional free ϵ chain complex C on p_X equipped with an n–dimensional ϵ quadratic structure is called an *n–dimensional ϵ quadratic R–module complex on p_X*.

Let $f\colon C \to D$ be a chain map between free chain complexes on p_X. An *$(n{+}1)$–dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on f* is a collection $\{\delta\psi_s, \psi_s | s \geq 0\}$ of geometric morphisms $\delta\psi_s\colon D^{n+1-r-s} \to D_r$, $\psi_s\colon C^{n-r-s} \to C_r$ $(r \in \mathbb{Z})$ of radius ϵ such that the following holds in addition to (*):

$$d(\delta\psi_s) + (-)^r(\delta\psi_s)d^* + (-)^{n-s}\big(\delta\psi_{s+1} + (-)^{s+1}T\delta\psi_{s+1}\big) + (-)^n f\psi_s f^* \sim_{3\epsilon} 0\colon D^{n-r-s} \to D_r,$$

for $s \geq 0$. An ϵ chain map $f\colon C \to D$ between an n–dimensional free ϵ chain complex C on p_X and an $(n{+}1)$–dimensional free ϵ chain complex D on p_X equipped with an $(n{+}1)$–dimensional ϵ quadratic structure is called an *$(n{+}1)$–dimensional ϵ quadratic R–module pair on p_X*. Obviously its *boundary* (C, ψ) is an n–dimensional ϵ quadratic R–module complex on p_X. We will suppress references to the coefficient ring R unless we need to emphasize the coefficient ring.

An ϵ *cobordism* of n–dimensional ϵ quadratic structures ψ on C and ψ' on C' is an $(n{+}1)$–dimensional ϵ quadratic structure $(\delta\psi, \psi \oplus -\psi')$ on some chain map $C \oplus C' \to D$. An ϵ *cobordism* of n–dimensional ϵ quadratic complexes (C, ψ), (C', ψ') on p_X is an $(n{+}1)$–dimensional ϵ quadratic pair on p_X

$$\big(\begin{pmatrix} f & f' \end{pmatrix}\colon C \oplus C' \to D, \quad (\delta\psi, \psi \oplus -\psi')\big)$$

with boundary $(C \oplus C', \psi \oplus -\psi')$. The union of adjoining cobordisms is defined using the formula in Ranicki [9]. The union of adjoining ϵ cobordisms is a 2ϵ cobordism.

ΣC and ΩC will denote the suspension and the desuspension of C respectively, and $\mathcal{C}(f)$ will denote the algebraic mapping cone of a chain map f.

Definition 2.1 Let W be a subset of X. An n–dimensional ϵ quadratic structure ψ on C is *ϵ Poincaré (over W)* if the algebraic mapping cone of the duality 3ϵ chain map

$$\mathcal{D}_\psi = (1+T)\psi_0\colon C^{n-*} \longrightarrow C$$

is 4ϵ contractible (over W). A quadratic complex (C, ψ) is *ϵ Poincaré (over W)* if ψ is ϵ Poincaré (over W). Similarly, an $(n{+}1)$–dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f\colon C \to D$ is *ϵ Poincaré (over W)* if the algebraic mapping cone of the duality 4ϵ chain map

$$\mathcal{D}_{(\delta\psi,\psi)} = \begin{pmatrix} (1+T)\delta\psi_0 \\ (-)^{n+1-r}(1+T)\psi_0 f^* \end{pmatrix}\colon D^{n+1-r} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

is 4ϵ contractible (over W) (or equivalently the algebraic mapping cone of the 4ϵ chain map

$$\overline{\mathcal{D}}_{(\delta\psi,\psi)} = ((1+T)\delta\psi_0 \quad f(1+T)\psi_0) \colon \mathcal{C}(f)^{n+1-*} \longrightarrow D$$

is 4ϵ contractible (over W)) and ψ is ϵ Poincaré (over W). A quadratic pair $(f,(\delta\psi,\psi))$ is *ϵ Poincaré (over W)* if $(\delta\psi,\psi)$ is ϵ Poincaré (over W). We will also use the notation $\mathcal{D}_{\delta\psi} = (1+T)\delta\psi_0$, although it does not define a chain map from D^{n+1-*} to D in general.

This definition is slightly different from the one given in Yamasaki [15] (especially when W is a proper subset of X). There a quadratic complex/pair was defined to be ϵ Poincaré over W if the duality map is an ϵ chain equivalence over W. If $\mathcal{C}(\mathcal{D}_\psi)$ (resp. $\mathcal{C}(\mathcal{D}_{(\delta\psi,\psi)})$) is 4ϵ contractible (over W), then $\mathcal{D}_\psi$ (resp. $\mathcal{D}_{(\delta\psi,\psi)}$) is only a "weak" 8ϵ chain equivalence over W.

Definition 2.2 A chain map $f\colon C \to D$ is a *weak ϵ chain equivalence over W* if

(1) f is an ϵ chain map;

(2) there exists a family $g = \{g_r\colon D_r \to C_r\}$ of geometric morphisms of radius ϵ such that
- dg_r and g_rd have radius ϵ, and
- $dg_r \sim_\epsilon g_{r-1}d$ over W

for all r;

(3) there exist two families $h = \{h_r\colon C_r \to C_{r+1}\}$ and $k = \{k_r\colon D_r \to D_{r+1}\}$ of ϵ morphisms such that
- $dh_r + h_{r-1}d \sim_{2\epsilon} 1 - g_rf_r$ over W, and
- $dk_r + k_{r-1}d \sim_{2\epsilon} 1 - f_rg_r$ over W.

for all r.

In other words a weak chain equivalence satisfies all the properties of a chain equivalence except that its inverse may not be a chain map outside of W.

Weak chain equivalences behave quite similarly to chain equivalences. For example, 2.3(3) and 2.4 of Ranicki [14] can be easily generalized as follows:

Proposition 2.3 *If $f\colon C \to D$ is a weak δ chain equivalence over V and $f'\colon D \to E$ is a weak ϵ chain equivalence over W, then $f'f$ is a weak $\delta+\epsilon$ chain equivalence over $V^{-\delta-\epsilon} \cap W^{-\delta}$. If we further assume that f is a δ chain equivalence, then $f'f$ is a weak $\delta+\epsilon$ chain equivalence over $V^{-\epsilon} \cap W^{-\delta}$.*

Proposition 2.4 *Let $f\colon C \to D$ be an ϵ chain map. If the algebraic mapping cone $\mathcal{C}(f)$ is ϵ contractible over W, then f is a weak 2ϵ chain equivalence over W. If f is a weak ϵ chain equivalence over W, then $\mathcal{C}(f)$ is 3ϵ contractible over $W^{-2\epsilon}$.*

We have employed the definition of Poincaré complexes/pairs using local contractibility of the algebraic mapping cone of the duality map, because algebraic mapping cones are easier to handle than chain equivalences. For example, consider a triad Γ:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, squiggly, "h"] & D \arrow[d, "g'"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd} \qquad dh + hd \sim f'g - g'f$$

and assume

(1) f (resp. f') is a δ (resp. δ') chain map,

(2) g (resp. g') is an ϵ (resp. ϵ') chain map,

(3) $h\colon g'f \simeq f'g$ is a γ chain homotopy.

Then there are induced a $\max\{\delta, \delta', 2\gamma\}$ chain map

$$F = \begin{pmatrix} f' & (-)^r h \\ 0 & -f \end{pmatrix} : \mathcal{C}(-g)_r = C'_r \oplus C_{r-1} \to \mathcal{C}(g')_r = D'_r \oplus D_{r-1}$$

and a $\max\{\epsilon, \epsilon', 2\gamma\}$ chain map

$$G = \begin{pmatrix} g' & (-)^r h \\ 0 & g \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f')_r = D'_r \oplus C'_{r-1}.$$

It is easily seen that $\mathcal{C}(F) = \mathcal{C}(G)$.

Proposition 2.5 *If $\mathcal{C}(g\colon C \to C')$ is ϵ contractible over W, then $\mathcal{C}(-g)$ is ϵ contractible over W.*

Proof Suppose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{C}(g)_r = C_r \oplus C'_{r-1} \longrightarrow \mathcal{C}(g)_{r+1} = C_{r+1} \oplus C'_r$$

is an ϵ chain contraction over W of $\mathcal{C}(g)$, then

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

is an ϵ chain contraction over W of $\mathcal{C}(-g)$. $\square$

Proposition 2.6 *Let Γ be as above, and further assume that $\mathcal{C}(g)$ is ϵ contractible over W and $\mathcal{C}(g')$ is ϵ' contractible over W, then $\mathcal{C}(G)$ is $3\max\{\epsilon,\epsilon',\delta,\delta',2\gamma\}$ contractible over $W^{-2\max\{\epsilon,\epsilon',\delta,\delta',2\gamma\}}$.*

Proof By Proposition 2.5, $\mathcal{C}(-g)$ is ϵ contractible over W. Therefore $F\colon \mathcal{C}(-g)\to\mathcal{C}(g')$ is a $\max\{\epsilon,\epsilon',\delta,\delta',2\gamma\}$ chain equivalence over W, and the proposition is proved by applying Proposition 2.4 to F. □

Corollary 2.7 *Let C and D be free ϵ chain complexes, and let $(\delta\psi,\psi)$ be an ϵ quadratic structure on an ϵ chain map $f\colon C\to D$. If $\mathcal{C}(\mathcal{D}_{(\delta\psi,\psi)})$ is 4ϵ contractible over W, then $\mathcal{C}(\mathcal{D}_\psi)$ is 100ϵ contractible over $W^{-100\epsilon}$.*

Proof Consider the triad Γ:

$$\begin{tikzcd} \Omega\mathcal{C}(f)^{n+1-*} \arrow[r, "\alpha=(1\ \ 0)"] \arrow[d, "\mathcal{D}_{(\delta\psi,\psi)}"'] \arrow[dr, squiggly, "h"] & \Omega\mathcal{D}^{(n+1-*)} \arrow[d, "\overline{\mathcal{D}}_{(\delta\psi,\psi)}"] \\ \Omega\mathcal{D} \arrow[r, "\beta={}^t(1\ \ 0)"'] & \Omega\mathcal{C}(f) \end{tikzcd} \qquad h=\begin{pmatrix}0 & 0\\ 0 & (-)^{r+1}(1+T)\psi_0\end{pmatrix}$$

and consider the chain map $G\colon \mathcal{C}(\alpha)\to\mathcal{C}(\beta)$ induced from Γ as above. Then $\mathcal{C}(G)$ is 12ϵ contractible over $W^{-8\epsilon}$ by the previous proposition. Therefore G is a weak 24ϵ chain equivalence over $W^{-8\epsilon}$. $(1+T)\psi_0$ is equal to the following composite of G with two ϵ chain equivalences:

$$C^{n-*}\xrightarrow[\simeq_\epsilon]{{}^t(0\ 0\ 1)}\mathcal{C}(\alpha)\xrightarrow{G}\mathcal{C}(\beta)\xrightarrow[\simeq_\epsilon]{(0\ 1\ 0)}C,$$

and the claim follows from Proposition 2.3. □

Next we describe various constructions on quadratic complexes with some size estimates. Firstly a direct calculation shows the following. (See the non-controlled case in Ranicki [9].)

Proposition 2.8 *If adjoining ϵ cobordisms c and c' are ϵ Poincaré over W, then $c\cup c'$ is 100ϵ Poincaré over $W^{-100\epsilon}$.*

The following proposition gives us a method to construct quadratic structures and cobordisms.

Proposition 2.9 *Suppose $g\colon C \to C'$ is a δ chain map of δ chain complexes and ψ is an n–dimensional ϵ quadratic structure on C.*

(1) *$g_{\%}\psi = \{(g_{\%}\psi)_s = g\psi_s g^*\}$ is a $2\delta+\epsilon$ quadratic structure on C', and $(0, \psi \oplus -g_{\%}\psi)$ is a $2\delta+\epsilon$ quadratic structure on the δ chain map $(g \quad 1)\colon C \oplus C' \longrightarrow C'$.*

(2) *If ψ is ϵ Poincaré over W and g is a weak δ chain equivalence over W, then $g_{\%}\psi$ and $(0, \psi \oplus -g_{\%}\psi)$ are $2\delta+6\epsilon$ Poincaré over $W^{-(6\delta+24\epsilon)}$.*

(3) *If ψ is ϵ Poincaré, g is a δ chain equivalence, $\Psi' = (\delta\psi, \psi' = g_{\%}\psi)$ is an ϵ' Poincaré δ' quadratic structure on a δ' chain map $f'\colon C' \to D$, and D is a γ chain complex, then $\Psi = (\delta\psi, \psi)$ is an $\epsilon' + 3\max\{9\delta, 6\delta', 4\epsilon, 3\gamma\}$ Poincaré $\max\{\delta, \epsilon\}$ quadratic structure on the $\delta'+\epsilon$ chain map $f = f' \circ g\colon C \to D$.*

Proof (1) This can be checked easily.

(2) This holds because the duality maps for $(C', g_{\%}\psi)$ and c split as follows:

$$(C')^{n-*} \xrightarrow{g^*} C^{n-*} \xrightarrow{(1+T)\psi_0} C \xrightarrow{g} C'$$
$$\mathcal{C}((g \quad 1))^{n+1-*} \xrightarrow[\simeq_\delta]{(0 \;\; 1 \;\; -g^*)} C^{n-*} \xrightarrow{(1+T)\psi_0} C \xrightarrow{g} C'$$

(3) We study the duality map for Ψ'. Since g is a δ chain equivalence and $\mathcal{C}(1\colon D \to D)$ is γ contractible, the algebraic mapping cone of the $\max\{\gamma, \delta\}$ chain map

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}\colon \mathcal{C}(f)_r = D_r \oplus C_{r-1} \longrightarrow \mathcal{C}(f') = D_r \oplus C'_{r-1}$$

is $3\max\{3\delta, \delta+\delta', \gamma\}$ contractible, and so is $\mathcal{C}(\tilde{g}^*\colon \mathcal{C}(f')^{n+1-*} \to \mathcal{C}(f)^{n+1-*})$. Therefore, the chain map $\mathcal{C}(\mathcal{D}_{\Psi'}) \to \mathcal{C}(\mathcal{D}_{\Psi})$ defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & \tilde{g}^* \end{pmatrix}\colon \mathcal{C}(\mathcal{D}_{\Psi'}) = D_r \oplus \mathcal{C}(f')^{n+2-r} \longrightarrow \mathcal{C}(\mathcal{D}_{\Psi}) = D_r \oplus \mathcal{C}(f)^{n+2-r}$$

is a $6\max\{9\delta, 6\delta', 4\epsilon, 3\gamma\}$ chain equivalence. The claim now follows from the next lemma. □

Lemma 2.10 *If a chain complex A is ϵ chain equivalent to a chain complex B which is δ contractible over $X - Y$, then A is $(2\epsilon+\delta)$ contractible over $X - Y^\epsilon$.*

Proof Let $f\colon A \to B$ be an ϵ chain equivalence, g an ϵ chain homotopy inverse, $h\colon gf \simeq_\epsilon 1$ an ϵ chain homotopy, and Γ a δ chain contraction of B over $X - Y$. Then $g\Gamma f + h$ gives a$(2\epsilon+\delta)$ chain contraction of A over $X - Y^\epsilon$. □

Remarks (1) An ϵ chain equivalence $f\colon\ C\longrightarrow C'$ such that $f_{\%}\psi = \psi'$ will be called an ϵ *homotopy equivalence* from (C,ψ) to (C',ψ'). By Proposition 2.9, a homotopy equivalence between quadratic Poincaré complexes induces a Poincaré cobordism between them.

(2) The estimates given in Proposition 2.9 and Lemma 2.10 are, of course, not acute in general. For example, consider an ϵ quadratic complex (C,ψ) which is ϵ Poincaré over W. Then a direct calculation shows that the cobordism between (C,ψ) and itself induced by the identity map of C is an ϵ quadratic pair and is ϵ Poincaré over W. This cobordism will be called the *trivial cobordism* from (C,ψ) to itself.

3 Epsilon-controlled L–groups

In this section we review the boundary construction of the first-named author and then introduce epsilon-controlled L–groups

$$L_n^{\delta,\epsilon}(X;p_X,R) \quad \text{and} \quad L_n^{\delta,\epsilon}(X,Y;p_X,R)$$

for $p_X\colon\ M\to X$, $Y\subset X$, $n\geq 0$, $\delta\geq\epsilon\geq 0$, and a ring R with involution. These are defined using geometric R–module chain complexes with quadratic Poincaré structures discussed in the previous section.

Let (C,ψ) be an n–dimensional ϵ quadratic R–module complex on p_X, where $n\geq 1$. Define a (possibly non-positive) 2ϵ chain complex ∂C by $\Omega\mathcal{C}(\mathcal{D}_\psi)$. Then an $(n-1)$–dimensional 2ϵ Poincaré 2ϵ quadratic structure $\partial\psi$ on ∂C is defined by:

$$\partial\psi_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} :\ \partial C^{n-r-1} = C^{n-r}\oplus C_{r+1}\longrightarrow \partial C_r = C_{r+1}\oplus C^{n-r}$$

$$\partial\psi_s = \begin{pmatrix} (-)^{n-r-s-1}T\psi_{s-1} & 0 \\ 0 & 0 \end{pmatrix} :$$

$$\partial C^{n-r-s-1} = C^{n-r-s}\oplus C_{r+s+1}\longrightarrow \partial C_r = C_{r+1}\oplus C^{n-r} \qquad (s\geq 1).$$

This is the *boundary construction* of Ranicki [9]. The structure $\Psi_1 = (0,\partial\psi)$ is an n–dimensional 2ϵ Poincaré 2ϵ quadratic structure on the ϵ chain map

$$i_C = \text{projection}\colon\ \partial C\longrightarrow C^{n-*}$$

of 2ϵ chain complexes. This is called the *algebraic Poincaré thickening* (see Ranicki [9]).

Example 3.1 Consider an n–dimensional ϵ chain complex F, and give ΣF the trivial $(n{+}1)$–dimensional quadratic structure $\theta_s = 0$ $(s \geq 0)$. Its algebraic Poincaré thickening

$$(i_{\Sigma F}\colon\ \partial\Sigma F = F \oplus F^{n-*} \longrightarrow (\Sigma F)^{n+1-*} = F^{n-*}, \quad (0, \partial\theta))$$

is an $(n{+}1)$–dimensional ϵ Poincaré ϵ null-cobordism of $(\partial\Sigma F, \partial\theta)$.

There is an inverse operation up to homotopy equivalence. Given an n–dimensional ϵ Poincaré ϵ quadratic pair $c = (f\colon\ C \to D, (\delta\psi, \psi))$, take the union $(\tilde{C}, \tilde{\psi})$ of c with the ϵ quadratic pair $(C \to 0, (0, -\psi))$. $\tilde{C}$ is equal to $\mathcal{C}(f)$. $(\tilde{C}, \tilde{\psi})$ is an n–dimensional 2ϵ quadratic complex and is called the *algebraic Thom complex* of c. The algebraic Poincaré thickening of $(\tilde{C}, \tilde{\psi})$ is "homotopy equivalent" to the original pair c (as pairs). Since we will not use this full statement, we do not define homotopy equivalences of pairs here and only mention that the chain map

$$g = \begin{pmatrix} 0 & 1 & 0 & -\psi_0 \end{pmatrix}\colon\ \partial\tilde{C}_r = D_{r+1} \oplus C_r \oplus D^{n-r} \oplus C^{n-r-1} \longrightarrow C_r$$

gives an 11ϵ chain equivalence such that $g_{\%}(\partial\tilde{\psi}) = \psi$. If we start with an n–dimensional ϵ quadratic complex (C, ψ) on p_X, then the algebraic Thom complex of the algebraic Poincaré thickening $(i_C\colon\ \partial C \longrightarrow C^{n-*}, (0, \partial\psi))$ of (C, ψ) is 3ϵ homotopy equivalent to (C, ψ); 3ϵ homotopy equivalences are given by

$$\begin{gathered} f = \begin{pmatrix} -\mathcal{D}_\psi & 1 & 0 \end{pmatrix}\colon\ \mathcal{C}(i_C)_r = C^{n-r} \oplus C_r \oplus C^{n-r+1} \longrightarrow C_r, \\ f_{\%}(\Psi_1 \cup_{\partial\psi} -\Psi_2) = \psi, \\ f' = {}^t\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\colon\ C_r \longrightarrow \mathcal{C}(i_C)_r = C^{n-r} \oplus C_r \oplus C^{n-r+1}, \\ f'_{\%}\psi = \Psi_1 \cup_{\partial\psi} -\Psi_2, \end{gathered}$$

where $\Psi_2 = (0, \partial\psi)$ is the n–dimensional ϵ quadratic structure on the trivial chain map $0\colon\ \partial C \longrightarrow 0$.

The boundary construction described above generalizes to quadratic pairs. For an $(n{+}1)$–dimensional ϵ quadratic pair $(f\colon\ C \to D, (\delta\psi, \psi))$ on p_X, define a (possibly non-positive) 2ϵ chain complex ∂D by $\Omega\mathcal{C}(\mathcal{D}_{(\delta\psi,\psi)})$ and define an n–dimensional 3ϵ Poincaré 2ϵ quadratic structure $\Psi_3 = (\partial\delta\psi, \partial\psi)$ on the 2ϵ chain map of 2ϵ chain complexes

$$\partial f = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\colon\ \partial C_r = C_{r+1} \oplus C^{n-r} \longrightarrow \partial D_r = D_{r+1} \oplus D^{n-r+1} \oplus C^{n-r}$$

by

$$\partial\delta\psi_0 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \partial D^{n-r} = D^{n-r+1} \oplus D_{r+1} \oplus C_r$$
$$\longrightarrow \partial D_r = D_{r+1} \oplus D^{n-r+1} \oplus C^{n-r}$$
$$\partial\delta\psi_s = \begin{pmatrix} (-)^{n-r-s-1} T\delta\psi_{s-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \partial D^{n-r-s} = D^{n-r-s+1} \oplus D_{r+s+1} \oplus C_{r+s}$$
$$\longrightarrow \partial D_r = D_{r+1} \oplus D^{n-r+1} \oplus C^{n-r} \qquad (s \geq 1).$$

$\partial\psi$ is the same as above. Then $(0, \Psi_1 \cup_{\partial\psi} (-\Psi_3))$ gives an $(n{+}1)$–dimensional 300ϵ Poincaré 4ϵ quadratic structure on the ϵ chain map

$$\begin{pmatrix} i & 0 & i_D \end{pmatrix} : (C^{n-*} \cup_{\partial C} \partial D)_r = C^{n-r} \oplus \partial C_{r-1} \oplus \partial D_r$$
$$\longrightarrow (\mathcal{C}(f)^{n+1-*})_r = \mathcal{C}(f)^{n+1-r}$$

of 2ϵ chain complexes, where $i\colon\ C^{n-*} \to \mathcal{C}(f)^{n+1-*}$ is the inclusion map and $i_D\colon\ \partial D \to \mathcal{C}(f)^{n+1-*}$ is the projection map.

If (C, ψ) (resp. $(f\colon\ C \to D, (\delta\psi, \psi))$) is ϵ Poincaré, then ∂C is (resp. ∂C and ∂D are) 4ϵ contractible, and hence chain homotopic to a positive chain complex (resp. positive chain complexes). But in general ∂C (and ∂D) may not be chain homotopic to a positive chain complex. This leads us to the following definition. The non-controlled version is described in Ranicki [9].

Definition 3.2 (1) A positive geometric chain complex C ($C_i = 0$ for $i < 0$) is ϵ *connected* if there exists a 4ϵ morphism $h\colon\ C_0 \to C_1$ such that $dh \sim_{8\epsilon} 1_{C_0}$.

(2) A chain map $f\colon\ C \to D$ of positive chain complexes is ϵ *connected* if $\mathcal{C}(f)$ is ϵ connected.

(3) A quadratic complex (C, ψ) is ϵ *connected* if $\mathcal{D}_\psi$ is ϵ connected.

(4) A quadratic pair $(f\colon\ C \to D, (\delta\psi, \psi))$ is ϵ *connected* if $\mathcal{D}_\psi$ and $\mathcal{D}_{(\delta\psi,\psi)}$ are ϵ connected.

Lemma 3.3 (1) *The composition of a δ connected chain map and an ϵ connected chain map is $\delta + \epsilon$ connected.*

(2) *Quadratic complexes and pairs that are ϵ Poincaré are ϵ connected.*

(3) *If ψ is an ϵ connected quadratic structure on C and $g\colon\ C \to C'$ is a δ connected chain map, then $\mathcal{D}_{(0,\psi\oplus -g_\%\psi)}$ is $\epsilon + 2\delta$ connected.*

Proof (1) is similar to Proposition 2.3. (2) is immediate from definition. (3) is similar to Proposition 2.9 (2). □

Remark In general the ϵ connectivity of g does not imply the ϵ connectivity of g^* (or δ connectivity for any δ). Therefore we do not have any estimate on the connectivity of $g_\%\psi$ in (3) above. It should be checked by an *ad hoc* method in each case. For example, see Section 6.

If the desuspension ΩC of a positive complex C on p_X is ϵ chain equivalent to a positive complex, then C is $\epsilon/4$ connected. On the other hand, we have:

Proposition 3.4 *Let $n \geq 1$.*

(1) *Suppose an n–dimensional ϵ quadratic complex (C, ψ) on p_X is ϵ connected. Then ∂C is 12ϵ chain equivalent to an $(n-1)$–dimensional (resp. a 1–dimensional) 4ϵ chain complex $\widehat{\partial} C$ if $n > 1$ (resp. if $n = 1$).*

(2) *Suppose an $(n+1)$–dimensional ϵ quadratic pair $(f\colon C \to D, (\delta\psi, \psi))$ is ϵ connected. Then ∂D is 24ϵ chain equivalent to an n–dimensional 5ϵ chain complex $\widehat{\partial} D$.*

(3) *When $n = 1$, the free 1–dimensional chain complex $(\widehat{\partial} C, 1)$ given in (1) and (2), viewed as a projective chain complex, is 32ϵ chain equivalent to a 0–dimensional 32ϵ projective chain complex $(\tilde{\partial} C, p)$ and there is a 32ϵ isomorphism*

$$(\widehat{\partial} C_1, 1) \oplus (\tilde{\partial} C_0, p) \longrightarrow (\widehat{\partial} C_0, 1),$$

and hence the controlled reduced projective class $\left[\widetilde{\partial} C, p\right]$ vanishes in $\widetilde{K}_0^{0,32\epsilon}(X; p_X, R)$.

Proof (1) There exists a 4ϵ morphism $h\colon \partial C_{-1} \to \partial C_0$ such that $dh \sim_{8\epsilon} 1$. Define a 4ϵ morphism $h'\colon \partial C_{n-1} \to \partial C_n$ by the composite:

$$h'\colon \partial C_{n-1} = C_n \oplus C^1 \xrightarrow{\begin{pmatrix} 0 & (-)^n \\ (-)^n & 0 \end{pmatrix}} C^1 \oplus C_n \xrightarrow{h^*} C^0 = \partial C_n,$$

then $h'd \sim_{8\epsilon} 1$. Now one can use the folding argument from the bottom (see Yamasaki [15]) using h and, if $n > 1$, from the top (see Ranicki–Yamasaki [14]) using h' to construct a desired chain equivalence.

(2) There exists a 4ϵ morphism $h\colon \partial D_{-1} \to \partial D_0$ such that $dh \sim_{8\epsilon} 1$. Define a 5ϵ morphism $h'\colon \partial D_n \to \partial D_{n+1}$ by the composite of

$$\begin{pmatrix} 0 & (-)^{n+1} & 0 \\ (-)^{n+1} & 0 & 0 \\ 0 & 0 & (1+T)\psi_0 \end{pmatrix} \colon \partial D_n = D_{n+1} \oplus D^1 \oplus C^0 \to \partial D^0 = D^1 \oplus D_{n+1} \oplus C_n$$

and $h^*\colon \partial D^0 \to \partial D^{-1} = \partial D_{n+1}$, then $h'd \sim_{8\epsilon} 1$. Use the folding argument again.

(3) The boundary map $\widehat{\partial}C_1 = \partial C_1 \oplus \partial C_{-1} \to \widehat{\partial}C_0 = \partial C_0$ is given by the matrix $(d_{\partial C} \quad h)$. Therefore

$$s = \begin{pmatrix} h' - h'hd_{\partial C} \\ d_{\partial C} \end{pmatrix} \colon \partial C_0 \longrightarrow \partial C_1 \oplus \partial C_{-1}$$

defines a 12ϵ morphism $s\colon \widehat{\partial}C_0 \to \widehat{\partial}C_1$ such that $sd_{\widehat{\partial}C} \sim_{16\epsilon} 1$. Define $\tilde{\partial}C_0$ to be $\widehat{\partial}C_0$ and define a 16ϵ morphism $p\colon \tilde{\partial}C_0 \to \tilde{\partial}C_0$ by $1 - d_{\widehat{\partial}C}$, then $p^2 \sim_{32\epsilon} p$ and $p\colon (\tilde{\partial}C_0, 1) \to (\tilde{\partial}C_0, p)$ defines the desired 32ϵ chain equivalence. The isomorphism can be obtained by combining the following isomorphisms:

$$(\widehat{\partial}C_1, 1) \underset{s}{\overset{d}{\rightleftarrows}} (\widehat{\partial}C_0, 1-p)$$

$$(\widehat{\partial}C_0, 1-p) \oplus (\widehat{\partial}C_0, p) \underset{{}^t(q \;\; p)}{\overset{(q \;\; p)}{\rightleftarrows}} (\widehat{\partial}C_0, 1)$$

This completes the proof. □

Controlled connectivity is preserved under union operation in the following manner.

Proposition 3.5 *If adjoining ϵ cobordisms c and c' are ϵ connected, then $c \cup c'$ is 100ϵ connected.*

Proof Similar to Proposition 2.8. □

Now we define the epsilon-controlled L–groups. Let Y be a subset of X.

Definition 3.6 For an integer $n \geq 0$, pair of non-negative numbers $\delta \geq \epsilon \geq 0$, and a ring with involution R, $L_n^{\delta,\epsilon}(X, Y; p_X, R)$ is defined to be the equivalence classes of finitely generated n–dimensional ϵ connected ϵ quadratic complexes on p_X that are ϵ Poincaré over $X - Y$. The equivalence relation is generated by finitely generated δ connected δ cobordisms that are δ Poincaré over $X - Y$.

Remark We use the following abbreviations for simplicity:

- $L_n^{\delta,\epsilon}(X; p_X, R) = L_n^{\delta,\epsilon}(X, \emptyset; p_X, R)$
- $L_n^{\epsilon}(X, Y; p_X, R) = L_n^{\epsilon,\epsilon}(X, Y; p_X, R)$
- $L_n^{\epsilon}(X; p_X, R) = L_n^{\epsilon,\epsilon}(X; p_X, R)$

Proposition 3.7 *Direct sum $(C,\psi)\oplus(C',\psi') = (C\oplus C',\psi\oplus\psi')$ induces an abelian group structure on $L_n^{\delta,\epsilon}(X,Y;p_X,R)$. Furthermore if $[C,\psi]=[C',\psi']\in L_n^{\delta,\epsilon}(X,Y;p_X,R)$, then there is a finitely generated 100δ connected 2δ cobordism between (C,ψ) and (C',ψ') that is 100δ Poincaré over $X-Y^{100\delta}$.*

Proof The inverse of an element $[C,\psi]$ is given by $[C,-\psi]$. In fact, as in Proposition 2.9 and Lemma 3.3 (with $g=1$),

$$((1\ \ 1)\colon\ C\oplus C\longrightarrow C,(0,\psi\oplus-\psi))$$

gives an ϵ connected ϵ null-cobordism of $(C,\psi)\oplus(C,-\psi)$ that is ϵ Poincaré over $X-Y$. The second claim follows from Proposition 2.8 and Proposition 3.5, because we can glue a sequence of cobordisms at once. □

If $\delta'\geq\delta$ and $\epsilon'\geq\epsilon$, then there is a homomorphism

$$L_n^{\delta,\epsilon}(X,Y;p_X,R)\longrightarrow L_n^{\delta',\epsilon'}(X,Y;p_X,R)$$

which sends $[C,\psi]$ to $[C,\psi]$. This is called the *relax-control map*.

In the study of controlled L–groups, we need an analogue of Proposition 2.9 for pairs:

Proposition 3.8 *Suppose there is a triad of ϵ chain complexes on p_X*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, squiggly, "g"] & D \arrow[d, "h"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd} \qquad dk+kd\sim f'g-hf$$

where f, f', g, h are ϵ chain maps and k is an ϵ chain homotopy, and suppose $(\delta\psi,\psi)$ is an $(n{+}1)$–dimensional ϵ quadratic structure on f.

(1) *There is induced a 4ϵ quadratic structure on f':*

$$\begin{aligned}(g,h;k)_\%(\delta\psi,\psi) = (&h\delta\psi_s h^* + (-)^{n+1}k\psi_s f^*h^* + (-)^{n-r+1}f'g\psi_s k^*\\ &+(-)^{r+1}kT\psi_{s+1}k^*\colon\ (D'^{n+1-r-s},q'^*)\to(D'_r,q'_r),g\psi_s g^*)_{s\geq0}.\end{aligned}$$

(2) *Suppose g and h are ϵ chain equivalences.*

(a) *If $(\delta\psi,\psi)$ is ϵ Poincaré over $X-Y$, then $(g,h;k)_\%(\delta\psi,\psi)$ is 30ϵ Poincaré over $X-Y^{81\epsilon}$.*

(b) *If $(f,(\delta\psi,\psi))$ is ϵ connected, then $(f',(g,h;k)_\%(\delta\psi,\psi))$ is 30ϵ connected.*

Proof (1) is easy to check. (2) can be checked by showing that

$$((-)^{n+1-r}k\psi_0 k^* \quad k(1+T)\psi_0 g^*)\colon\ \mathcal{C}(f')^{n+1-r} = D'^{n+1-r} \oplus C'^{n-r} \longrightarrow D'_{r+1}$$

is a 3ϵ chain homotopy between the duality map for $(g,h;k)_{\%}(\delta\psi,\psi)$ and the following chain map:

$$h((1+T)\delta\psi_0 \quad f(1+T)\psi_0)\begin{pmatrix} h^* & 0 \\ (-)^{n+1-r}k^* & g^* \end{pmatrix}\colon\ \mathcal{C}(f')^{n+1-r} \longrightarrow (D'_r, q'_r),$$

which is a weak 27ϵ chain equivalence over $X - Y^{18\epsilon}$ in case (2a), and 16ϵ connected in case (2b). □

Corollary 3.9 *Suppose $f\colon C \to D$ is an ϵ chain map, $(\delta\psi,\psi)$ is an $(n{+}1)$–dimensional ϵ quadratic structure on f, $g\colon C \to C'$ is a γ chain equivalence, and $h\colon D \to D'$ is a δ chain equivalence. Let $\epsilon' = \gamma + \delta + \epsilon$ and g^{-1} be a γ chain homotopy inverse of g.*

(1) *There is an $(n{+}1)$–dimensional $4\epsilon'$ quadratic structure $(\delta\psi', \psi' = g_{\%}\psi)$ on the ϵ' chain map $f' = hfg^{-1}\colon (C',p') \to (D',q')$.*

(2) *If $(\delta\psi,\psi)$ is ϵ Poincaré over $X - Y$, then $(\delta\psi',\psi')$ is $30\epsilon'$ Poincaré over $X - Y^{81\epsilon'}$.*

(3) *If $(\delta\psi,\psi)$ is ϵ connected, then $(\delta\psi',\psi')$ is $30\epsilon'$ connected.*

Proof Let $\Gamma\colon g^{-1}g \simeq 1$ be a γ chain homotopy. Define an ϵ' chain homotopy $k\colon hf \simeq f'g$ by $k = -hf\Gamma$, and apply Proposition 3.8 □

The last topic of this section is the functoriality. A *map between control maps* $p_X\colon M \to X$ and $p_Y\colon N \to Y$ means a pair of continuous maps $(f\colon M \to N, \bar{f}\colon X \to Y)$ which makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[\bar{f}]{} & Y. \end{array}$$

For example, given a control map $p_Y\colon N \to Y$ and a subset $X \subset Y$, let us denote the control map $p_Y|p_Y^{-1}(X)\colon p_Y^{-1}(X) \to X$ by $p_X\colon M \to X$. Then the inclusion maps $j\colon M \to N$, $\bar{\jmath}\colon X \to Y$ form a map form p_X to p_Y.

Epsilon controlled L–groups are functorial with respect to maps and relaxation of control in the following sense.

Proposition 3.10 *Let $F = (f, \bar{f})$ be a map from $p_X\colon M \to X$ to $p_Y\colon N \to Y$, and suppose that $\bar{f}$ is Lipschitz continuous with Lipschitz constant λ, ie there exists a constant $\lambda > 0$ such that*

$$d(\bar{f}(x_1), \bar{f}(x_2)) \leq \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

Then F induces a homomorphism

$$F_*\colon\ L_n^{\delta,\epsilon}(X, X'; p_X, R) \longrightarrow L_n^{\delta',\epsilon'}(Y, Y'; p_Y, R)$$

if $\delta' \geq \lambda\delta$, $\epsilon' \geq \lambda\epsilon$ and $\bar{f}(X') \subset Y'$. If two maps $F = (f, \bar{f})$ and $G = (g, \bar{g})$ are homotopic through maps $H_t = (h_t, \bar{h}_t)$ such that each $\bar{h}_t$ is Lipschitz continuous with Lipschitz constant λ, $\delta' > \lambda\delta$, $\epsilon' \geq \lambda\epsilon$, and $\bar{h}_t(X') \subset Y'$, then F and G induce the same homomorphism:

$$F_* = G_*\colon\ L_n^{\delta,\epsilon}(X, X'; p_X, R) \longrightarrow L_n^{\delta',\epsilon'}(Y, Y'; p_Y, R).$$

Proof The direct image construction for geometric modules and morphisms (see Ranicki–Yamasaki [14, page 7]) can be used to define the direct images $f_\#(C, \psi)$ of quadratic complexes and the direct images of cobordisms. And this induces the desired F_*.

For the second part, split the homotopy into thin layers to construct small cobordisms. The size of the cobordism may be slightly bigger than the size of the object itself. □

Remark The above is stated for Lipschitz continuous maps to simplify the statement. For specific $\delta \geq \epsilon$ and $\delta' \geq \epsilon'$, the following condition, instead of the Lipschitz condition above, is sufficient for the existence of F_*:

$$\begin{aligned} d(\bar{f}(x_1), \bar{f}(x_2)) &\leq k\epsilon' \quad \text{whenever} \quad d(x_1, x_2) \leq k\epsilon, \text{ and} \\ d(\bar{f}(x_1), \bar{f}(x_2)) &\leq k\delta' \quad \text{whenever} \quad d(x_1, x_2) \leq k\delta, \end{aligned}$$

for $k = 1, 3, 4, 8$. The second part of the proposition also holds under this condition. When X is compact and $\delta' \geq \epsilon'$ are given, the uniform continuity of $\bar{f}$ implies that this condition is satisfied for sufficiently small pairs $\delta \geq \epsilon$.

4 Epsilon-controlled projective L–groups

Fix a subset Y of X, and let $\mathcal{F}$ be a family of subsets of X such that $Z \supset Y$ for each $Z \in \mathcal{F}$. In this section we introduce intermediate epsilon-controlled L–groups $L_n^{\mathcal{F},\delta,\epsilon}(Y; p_X, R)$, which will appear in the stable-exact sequence of a pair (Section 5) and

also in the Mayer–Vietoris sequence (Section 7). Roughly speaking, these are defined using "controlled projective quadratic chain complexes" $((C,p),\psi)$ with vanishing ϵ–controlled reduced projective class $[C,p]=0\in\widetilde{K}_0^{n,\epsilon}(Z;p_Z,R)$ for each $Z\in\mathcal{F}$.

$\widetilde{K}_0^{n,\epsilon}(Z;p_Z,R)$ is an abelian group defined as the set of equivalence classes $[C,p]$ of finitely generated ϵ projective chain complexes on p_Z. See Ranicki–Yamasaki [14] for the details. The following is known [14, 3.1 and 3.5]:

Proposition 4.1 *If $[C,p]=0\in\widetilde{K}_0^{n,\epsilon}(Z;p_Z,R)$, then there is an n–dimensional free ϵ chain complex $(E,1)$ such that $(C,p)\oplus(E,1)$ is 3ϵ chain equivalent to an n–dimensional free ϵ chain complex on p_Z. If we further assume that $n\geq 1$, then (C,p) itself is 60ϵ chain equivalent to an n–dimensional free 30ϵ chain complex on p_Z.*

All the materials in the previous two sections (except for Proposition 3.4(3)) have obvious analogues in the category of projective chain complexes with the identity maps in the formulae replaced by appropriate projections. So we shall only describe the basic definitions and omit stating the obvious analogues of Propositions 2.3–2.6, Corollary 2.7, Propositions 2.8 and 2.9, Lemmas 2.10 and 3.3, Propositions 3.5 and 3.8, and Corollary 3.9, and we refer them by Proposition 2.3′, Proposition 2.4′, ... An analogue of Proposition 3.7 will be explicitly stated in Proposition 4.4 below.

For a projective module (A,p) on p_X, its dual $(A,p)^*$ is the projective module (A^*,p^*) on p_X. If $f\colon (A,p)\to(B,q)$ is an ϵ morphism [14], then $f^*\colon (B,q)^*\to(A,p)^*$ is also an ϵ morphism. For an ϵ projective chain complex on p_X

$$(C,p)\colon\ \ldots\longrightarrow(C_r,p_r)\xrightarrow{d_r}(C_{r-1},p_{r-1})\xrightarrow{d_{r-1}}\ldots$$

in the sense of [14], $(C,p)^{n-*}$ will denote the ϵ projective chain complex on p_X defined by:

$$\ldots\longrightarrow(C^{n-r},p^*_{n-r})\xrightarrow{(-)^r d_r^*}(C^{n-r+1},p^*_{n-r+1})\longrightarrow\ldots.$$

Before we go on to define ϵ projective quadratic complexes, we need to define basic notions for projective chain complexes. For $Y=X$ or for free chain complexes, these are already defined in [14].

Suppose $f\colon (A,p)\to(B,q)$ is a morphism between projective modules on p_X, and let Y be a subset of X. The *restriction* $f|Y$ of f to Y will mean the restriction of f in the sense of [14, page 21] with f viewed as a geometric morphism from A to B; that is, $f|Y$ is the sum of the paths (with coefficients) that start from points in $p_X^{-1}(Y)$. $f|Y$ can be viewed as a geometric morphism from A to B and also as a geometric morphism from $A(Y)$ to $B(Y)$, where $A(Y)$ denotes the restriction of A to Y in the sense of [14], ie the

geometric submodule of A generated by the basis elements of A that are in $p_X^{-1}(Y)$. But, in general, it does not give a morphism from (A,p) to (B,q). Also note that there is no obvious way to "restrict" a projection $p\colon A \to A$ to a projection on $A(Y)$.

The following four paragraphs are almost verbatim copies of the definitions for free chain complexes [14, page 22].

Let $f,\ g\colon (A,p) \to (B,q)$ be morphisms; f is said to be *equal to g over Y* ($f = g$ *over* Y) if $f|Y = g|Y$, and f is said to be ϵ *homotopic to g over Y* ($f \sim_\epsilon g$ *over* Y) if $f|Y \sim_\epsilon g|Y$.

Let $f,\ g\colon (C,p) \to (D,q)$ be chain maps between projective chain complexes on p_X. A collection $\{h_r\colon (C_r,p_r) \to (D_{r+1},q_{r+1})\}$ of ϵ morphisms is said to be an ϵ *chain homotopy over Y between f and g* if $dh + hd \sim_{2\epsilon} g - f$ over Y.

An ϵ chain map $f\colon (C,p) \to (D,q)$ is said to be an ϵ *chain equivalence over Y* if there exist an ϵ chain map $g\colon (D,q) \to (C,p)$ and ϵ chain homotopies over Y between gf and p and between fg and q.

A chain complex (C,p) is said to be ϵ *contractible over Y* if there is an ϵ chain homotopy over Y between $0\colon (C,p) \to (C,p)$ and $p\colon (C,p) \to (C,p)$; such a chain homotopy over Y is called an ϵ *chain contraction of (C,p) over Y*.

The Definition 2.2 of *weak ϵ chain equivalences over Y* (for chain maps between free chain complexes) can be rewritten for maps between projective chain complexes in the obvious manner.

The following is the most important technical proposition in the theory of controlled projective chain complexes.

Proposition 4.2 (Ranicki–Yamasaki [14, 5.1 and 5.2]) *If an n–dimensional free ϵ chain complex C on p_X is ϵ contractible over $X - Y$, then $(C,1)$ is $(6n+15)\epsilon$ chain equivalent to an n–dimensional $(3n+12)\epsilon$ projective chain complex on $p_{Y^{(4n+14)\epsilon}}$. Conversely, if an n–dimensional free chain complex $(C,1)$ on p_X is ϵ chain equivalent to a projective chain complex (D,r) on p_Y, then C is ϵ contractible over $X - Y^\epsilon$.*

Now we introduce quadratic structures on projective chain complexes and pairs. An *n–dimensional ϵ quadratic structure* on a projective chain complex (C,p) on p_X is an n–dimensional ϵ quadratic structure ψ on C (in the sense of Section 2) such that $\psi_s\colon (C^{n-r-s},p^*) \to (C_r,p)$ is an ϵ morphism for every $s \geq 0$ and $r \in \mathbb{Z}$. Similarly, an *$(n{+}1)$–dimensional ϵ quadratic structure* on a chain map $f\colon (C,p) \to (D,q)$ is an $(n{+}1)$–dimensional ϵ quadratic structure $(\delta\psi,\psi)$ on $f\colon C \to D$ such that

$\delta\psi_s\colon (D^{n+1-r-s}, q^*) \to (D_r, q)$ and $\psi_s\colon (C^{n-r-s}, p^*) \to (C_r, p)$ are ϵ morphisms for every $s \geq 0$ and $r \in \mathbb{Z}$. An n–dimensional ϵ projective chain complex (C, p) on p_X equipped with an n–dimensional ϵ quadratic structure is called an *n–dimensional ϵ projective quadratic complex on p_X*, and an ϵ chain map $f\colon (C, p) \to (D, q)$ between an n–dimensional ϵ projective chain complex (C, p) on p_X and an $(n{+}1)$–dimensional ϵ projective chain complex (D, q) on p_X equipped with an $(n{+}1)$–dimensional ϵ quadratic structure is called an *$(n{+}1)$–dimensional ϵ projective quadratic pair on p_X*.

An ϵ *cobordism* of n–dimensional ϵ projective quadratic complexes $((C, p), \psi)$, $((C', p'), \psi')$ on p_X is an $(n{+}1)$–dimensional ϵ projective quadratic pair on p_X

$$((f\ \ f')\colon (C, p) \oplus (C', p') \longrightarrow (D, q), (\delta\psi, \psi \oplus -\psi'))$$

with boundary $((C, p) \oplus (C', p'), \psi \oplus -\psi')$.

Boundary constructions, algebraic Poincaré thickenings, algebraic Thom complexes, ϵ connectedness are defined as in the previous section.

An n–dimensional ϵ quadratic structure ψ on (C, p) is ϵ *Poincaré (over Y)* if

$$\partial(C, p) = \Omega\mathcal{C}((1 + T)\psi_0\colon (C^{n-*}, p^*) \longrightarrow (C, p))$$

is 4ϵ contractible (over Y). $((C, p), \psi)$ is ϵ *Poincaré (over Y)* if ψ is ϵ Poincaré (over Y). Similarly, an $(n{+}1)$–dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f\colon (C, p) \to (D, q)$ is ϵ *Poincaré (over Y)* if $\partial(C, p)$ and

$$\partial(D, q) = \Omega\mathcal{C}(((1 + T)\delta\psi_0 \quad f(1 + T)\psi_0)\colon \mathcal{C}(f)^{n+1-*} \longrightarrow (D, q))$$

are both 4ϵ contractible (over Y). A pair $(f, (\delta\psi, \psi))$ is ϵ *Poincaré (over Y)* if $(\delta\psi, \psi)$ is ϵ Poincaré (over Y).

Let Y and be a subset of X and $\mathcal{F}$ be a family of subsets of X such that $Z \supset Y$ for every $Z \in \mathcal{F}$.

Definition 4.3 Let $n \geq 0$ and $\delta \geq \epsilon \geq 0$. $L_n^{\mathcal{F},\delta,\epsilon}(Y; p_X, R)$ is the equivalence classes of finitely generated n–dimensional ϵ Poincaré ϵ projective quadratic complexes $((C, p), \psi)$ on p_Y such that $[C, p] = 0$ in $\widetilde{K}_0^{n,\epsilon}(Z; p_Z, R)$ for each $Z \in \mathcal{F}$. The equivalence relation is generated by finitely generated δ Poincaré δ cobordisms $((f\ \ f')\colon (C, p) \oplus (C', p') \to (D, q), (\delta\psi, \psi \oplus -\psi'))$ on p_Y such that $[D, q] = 0$ in $\widetilde{K}_0^{n+1,\delta}(Z; p_Z, R)$ for each $Z \in \mathcal{F}$.

Remark We use the following abbreviation: $L_n^{\mathcal{F},\epsilon}(Y; p_X, R) = L_n^{\mathcal{F},\epsilon,\epsilon}(Y; p_X, R)$.

Proposition 4.4 *Direct sum induces an abelian group structure on* $L_n^{\mathcal{F},\delta,\epsilon}(Y;p_X,R)$*. Furthermore if* $[(C,p),\psi] = [(C',p'),\psi'] \in L_n^{\mathcal{F},\delta,\epsilon}(Y;p_X,R)$*, then there is a finitely generated* 100δ *Poincaré* 2δ *cobordism on* p_Y

$$((f\ f')\colon (C,p)\oplus(C',p')\to(D,q),(\delta\psi,\psi\oplus-\psi'))$$

such that $[D,q]=0$ *in* $\widetilde{K}_0^{n+1,9\delta}(Z;p_Z,R)$ *for each* $Z\in\mathcal{F}$*.*

Proof The first part is similar to the proof of Proposition 3.7. Observe that $[D,q]=0$ in $\widetilde{K}_0^{n+1,9\delta}(Z;p_Z,R)$, because

$$[\mathcal{C}(g\colon (E,r)\to(F,s))] = [F,s]-[E,r]\in\widetilde{K}_0^{n+1,9\delta}(Z;p_Z,R)$$

for any δ chain map g between δ projective chain complexes (E,r) (of dimension n) and (F,s) (of dimension $n+1$) on p_Z. See Ranicki–Yamasaki [14, page 18]. □

A functoriality with respect to maps and relaxation of control similar to Proposition 3.10 holds for epsilon-controlled projective L–groups.

Proposition 4.5 *Let* $F=(f,\bar{f})$ *be a map from* $p_X\colon M\to X$ *to* $p_Y\colon N\to Y$*, and suppose that* $\bar{f}$ *is Lipschitz continuous with Lipschitz constant* λ*, ie there exists a constant* $\lambda>0$ *such that*

$$d(\bar{f}(x_1),\bar{f}(x_2))\le\lambda d(x_1,x_2)\qquad(x_1,x_2\in X).$$

If $\delta'\ge\lambda\delta$*,* $\epsilon'\ge\lambda\epsilon$*,* $\bar{f}(A)\subset B$*, and there exists a* $Z\in\mathcal{F}$ *satisfying* $\bar{f}(Z)\subset Z'$ *for each* $Z'\in\mathcal{F}'$*, then* F *induces a homomorphism*

$$F_*\colon\ L_n^{\mathcal{F},\delta,\epsilon}(A;p_X,R)\longrightarrow L_n^{\mathcal{F}',\delta',\epsilon'}(B;p_Y,R).$$

It is λ*–Lipschitz-homotopy invariant if* $\delta'>\lambda\delta$ *in addition.*

Remark As in the remark to Proposition 3.10, for a specific δ and ϵ, we do not need the full Lipschitz condition to guarantee the existence of F_*.

There is an obvious homomorphism

$$\iota\colon\ L_n^{\delta,\epsilon}(Y;p_Y,R)\longrightarrow L_n^{\mathcal{F},\delta,\epsilon}(Y;p_X,R);\qquad [C,\psi]\mapsto[(C,1),\psi]$$

from free L–groups to projective L–groups. On the other hand, the controlled K–theoretic condition posed in the definition can be used to construct homomorphisms from projective L–groups to free L–groups:

Proposition 4.6 *There exist a constant $\alpha > 1$ such that the following holds true: for any control map $p_X\colon M \to X$, any subset $Y \subset X$, any family of subsets $\mathcal{F}$ of X containing Y, any element Z of $\mathcal{F}$, any number $n \geq 0$, and any pair of positive numbers $\delta \geq \epsilon$ and $\delta \geq \epsilon$ with $\delta' \geq \alpha\delta$, $\epsilon' \geq \alpha\epsilon$, there is a well-defined homomorphism*

$$\iota_Z\colon\ L_n^{\mathcal{F},\delta,\epsilon}(Y;p_X,R) \longrightarrow L_n^{\delta',\epsilon'}(Z;p_Z,R),$$

functorial with respect to relaxation of control, such that the composite maps

$$L_n^{\mathcal{F},\delta,\epsilon}(Y;p_X,R) \xrightarrow{\iota_Z} L_n^{\delta',\epsilon'}(Z;p_Z,R) \xrightarrow{\iota} L_n^{p,\delta',\epsilon'}(Z;p_Z,R)$$
$$L_n^{\delta,\epsilon}(Y;p_Y,R) \xrightarrow{\iota} L_n^{\mathcal{F},\delta,\epsilon}(Y;p_X,R) \xrightarrow{\iota_Z} L_n^{\delta',\epsilon'}(Z;p_Z,R)$$

are equal to the ones induced from inclusion maps.

Remark Actually $\alpha = 20000$ works. In the rest of the paper, we always assume that $\alpha = 20000$.

Proof Let $[(C,p),\psi]$ be an element of $L_n^{\mathcal{F},\delta,\epsilon}(Y;p_X,R)$, and fix $Z \in \mathcal{F}$. Recall that $[C,p] = 0 \in \widetilde{K}_0^{n,\epsilon}(Z;p_Z,R)$. By Proposition 4.1, there exists an n–dimensional free ϵ chain complex $(E,1)$ on p_Z such that $(C,p)\oplus(E,1)$ is 3ϵ chain equivalent to some n–dimensional free ϵ chain complex $(\overline{F},1)$ on p_Z. Add $1\colon (E^{n-*},1) \to (E^{n-*},1)$ to this chain equivalence to get a 3ϵ chain equivalence

$$g\colon\ (C,p)\oplus(\partial\Sigma E,1) \longrightarrow (\overline{F},1)\oplus(E^{n-*},1) = (F,1)$$

of projective chain complexes on p_Z, where ΣE is defined using the trivial $(n{+}1)$–dimensional quadratic structure $\theta = 0$ on ΣE. See Example 3.1. We set

$$\iota_Z[(C,p),\psi] = [F, g_{\%}(\psi \oplus \partial\theta)].$$

Let us show that this defines a well-defined map. Suppose $[(C,p),\psi] = [(C',p'),\psi']$ in $L_n^{\mathcal{F},\delta,\epsilon}(Y;p_X,R)$, and let E and E' be n–dimensional free ϵ chain complexes on p_Z together with 3ϵ chain equivalences

$$g\colon\ (\overline{C},\overline{p}) = (C,p)\oplus(\partial\Sigma E,1) \to (F,1)$$
$$g'\colon\ (\overline{C}',\overline{p}') = (C',p')\oplus(\partial\Sigma E',1) \to (F',1)$$

to free ϵ chain complexes F and F' on p_Z. By Proposition 4.4 above and Proposition 4.1, there is a 100δ Poincaré 2δ null-cobordism

$$(f\colon\ (C,p)\oplus(C',p') \longrightarrow (D,q), (\delta\psi, \psi\oplus -\psi'))$$

such that (D, q) is 540δ chain equivalent to an $(n{+}1)$–dimensional free 270δ chain complex $(G, 1)$ (as a projective chain complex on p_Z). Take the direct sum with the null-cobordisms

$$(i_{\Sigma E}\colon\ (\partial \Sigma E, 1) \longrightarrow (E^{n-*}, 1), (0, \partial\theta)),$$
$$(i_{\Sigma E'}\colon\ (\partial \Sigma E', 1) \longrightarrow (E'^{n-*}, 1), (0, -\partial\theta')).$$

Now the claim follows from Corollary 3.9$'$.

$$\begin{tikzcd}
(\overline{C}, \overline{p}) \oplus (\overline{C}', \overline{p}') \arrow[r, "100\delta \text{ Poincaré}"] \arrow[d, "g\oplus g'"', "\simeq_{3\epsilon}"] & (D, q) \oplus (E^{n-*}, 1) \oplus (E'^{n-*}, 1) \arrow[d, "\simeq_{540\delta}"] \\
(F, 1) \oplus (F', 1) \arrow[r, dotted, "20000\delta \text{ Poincaré}"'] & (G, 1) \oplus (E^{n-*}, 1) \oplus (E'^{n-*}, 1)
\end{tikzcd}$$

This completes the proof. $\square$

5 Stably-exact sequence of a pair

Let Y be a subset of X. We discuss relations between the various controlled L–groups of X, Y, and (X, Y) by fitting them into a stably-exact sequence. Two of the three kinds of maps forming the sequence have already appeared. The first is the map

$$i_* = \iota_X\colon\ L_n^{\{X\},\delta,\epsilon}(Y; p_X, R) \longrightarrow L_n^{\delta',\epsilon'}(X; p_X, R)$$

defined when $\delta' \geq \alpha\delta$ and $\epsilon' \geq \alpha\epsilon$. The second is the homomorphism induced by the inclusion map $j\colon\ (X, \emptyset) \to (X, Y)$:

$$j_*\colon\ L_n^{\delta,\epsilon}(X; p_X, R) \to L_n^{\delta',\epsilon'}(X, Y; p_X, R).$$

defined for positive numbers $\delta' \geq \delta$ and $\epsilon' \geq \epsilon$. The third map ∂ is described in the next proposition.

Proposition 5.1 *For $n \geq 1$, there exists a constant $\kappa_n > 1$ such that the following holds true: If $Y' \supset Y^{\kappa_n\delta}$, $\delta' \geq \kappa_n\delta$, and $\epsilon' \geq \kappa_n\epsilon$, $\partial([C, \psi]) = [(E, q), \beta_{\%}\partial\psi]$ defines a well-defined homomorphism:*

$$\partial\colon\ L_n^{\delta,\epsilon}(X, Y; p_X, R) \to L_{n-1}^{\{X\},\delta',\epsilon'}(Y'; p_X, R),$$

where

$$\beta\colon\ (\partial C, 1) \longrightarrow (E, q)$$

is any $(200n + 300)\epsilon$ chain equivalence from $(\partial C, 1)$ to some $(n{-}1)$–dimensional $(100n + 300)\epsilon$ projective chain complex on $p_{Y'}$.

Remark Actually $\kappa_n = 150000(n+2)$ works. In the rest of the paper, we always assume that $\kappa_n \geq 150000(n+2)$.

Proof We first show the existence of such β. Take $[C,\psi] \in L_n^{\delta,\epsilon}(X,Y;p_X,R)$. Suppose $n>1$. By Proposition 3.4(1), there is a 12ϵ chain equivalence between ∂C and an $(n-1)$–dimensional 4ϵ chain complex $\widehat{\partial}C$ on p_X. Since ∂C is 4ϵ contractible over $X-Y$, $\widehat{\partial}C$ is 28ϵ contractible over $X-Y^{12\epsilon}$ by Lemma 2.10. Now by Proposition 4.2, $(\widehat{\partial}C,1)$ is $(168n+252)\epsilon$ $(=(6(n-1)+15)\times 28\epsilon)$ chain equivalent to an $(n-1)$–dimensional $(84n+252)\epsilon$ projective chain complex on $p_{Y^{(112n+292)\epsilon}}$.

Next suppose $n=1$. By Proposition 3.4(1) and (3), there is a 44ϵ chain equivalence between $(\partial C,1)$ and a 0–dimensional 32ϵ chain complex $(\tilde{\partial}C,p)$. Since ∂C is 4ϵ contractible over $X-Y$, $(\tilde{\partial}C,p)$ is 92ϵ contractible over $X-Y^{44\epsilon}$, ie $p\sim_{184\epsilon}0$ over $X-Y^{44\epsilon}$. Let $E=\tilde{\partial}C|Y^{76\epsilon}$ and $q=p|Y^{44\epsilon}$, then $p-q=p|(X-Y^{44\epsilon})\sim_{184\epsilon}0$. Therefore

$$q\ \sim_{184\epsilon}\ p\ \sim_{32\epsilon}\ p^2\ \sim_{216\epsilon}\ pq\ \sim_{216\epsilon}\ q^2,$$

and (E,q) is a 0–dimensional 216ϵ projective chain complex on $p_{Y^{292\epsilon}}$. The 32ϵ morphism q defines a 216ϵ isomorphism between $(\tilde{\partial}C,p)$ and (E,q) in each direction. Therefore $(\partial C,1)$ is 260ϵ chain equivalent to (E,q). This completes the proof of the existence of β.

Suppose $[C,\psi]=[C'\psi']\in L_n^{\delta,\epsilon}(X,Y;p_X,R)$ and suppose $\beta\colon (\partial C,1)\to(E,q)$ and $\beta'\colon (\partial C',1)\to(E',q')$ are chain equivalences satisfying the condition, and suppose Y', δ', and ϵ' satisfy the hypothesis. We show that $((E,q),\beta_\%\partial\psi)$ and $((E',q'),\beta'_\%\partial\psi')$ represent the same element in $L_{n-1}^{\{X\},\delta',\epsilon'}(Y';p_X,R)$. Without loss of generality, we may assume that there is an ϵ connected ϵ cobordism

$$((f\quad f')\colon\ C\oplus C'\longrightarrow D,\ (\partial\psi,\psi\oplus -\psi'))$$

which is ϵ Poincaré over $X-Y$. Apply the boundary construction (Section 3)to this pair to get a 3ϵ Poincaré 2ϵ quadratic structure $(\partial\delta\psi,\partial\psi\oplus-\partial\psi')$ on the 2ϵ chain map $(\partial C,1)\oplus(\partial C',1)\longrightarrow(\partial D,1)$ of 2ϵ chain complexes. By Lemma 2.10$'$, Proposition 3.4 and Proposition 4.2, $(\partial D,1)$ is $(312n+904)\epsilon$ chain equivalent to an n–dimensional $(156n+624)\epsilon$ projective chain complex (F,r) on $p_{Y^{(208n+752)\epsilon}}$. Now, by Corollary 3.9$'$, we can obtain a $(15360n+36210)\epsilon$ Poincaré cobordism

$$(E,q)\oplus(E',q')\longrightarrow(F,r),\qquad (\chi,\beta_\%\partial\psi\oplus(-\beta'_\%\partial\psi')).$$

Since such a structure involves $8(15360n+36210)\epsilon$ homotopies, this cobordism can be viewed to be on $p_{Y^{(123088n+290432)\epsilon}}$. Also $[F,r]=[\widehat{\partial}D,1]=0$ in $\widetilde{K}_0^{n,\epsilon'}(X;p_X,R)$,

and similarly $[E,q]=[E',q']=0$ in $\widetilde{K}_0^{n-1,\epsilon'}(X;p_X,R)$. Therefore $[(E,q),\beta_{\%}\partial\psi]=[(E',q'),\beta'_{\%}\partial\psi']$ in $L_{n-1}^{\{X\},\delta',\epsilon'}(Y';p_X,R)$.

□

Theorem 5.2 *For any integer $n\geq 0$, there exists a constant $\lambda_n>1$ which depends only on n such that the following holds true for any control map p_X and a subset Y of X:*

(1) *Suppose $\delta'\geq\alpha\delta$, $\epsilon'\geq\alpha\epsilon$, $\delta''\geq\delta'$, and $\epsilon''\geq\epsilon'$ so that the following two maps are defined:*

$$L_n^{\{X\},\delta,\epsilon}(Y;p_X,R)\xrightarrow{i_*}L_n^{\delta',\epsilon'}(X;p_X,R)\xrightarrow{j_*}L_n^{\delta'',\epsilon''}(X,W;p_X,R).$$

If $W\supset Y^{\alpha\epsilon}$, then j_*i_* is zero.

(2) *Suppose $\delta''\geq\delta'$, $\epsilon''\geq\epsilon'$ so that $j_*\colon L_n^{\delta',\epsilon'}(X;p_X,R)\to L_n^{\delta'',\epsilon''}(X,W;p_X,R)$ is defined. If $\delta\geq\lambda_n\delta''$ and $Y\supset W^{\lambda_n\delta''}$, then the relax-control image of the kernel of j_* in $L_n^{\alpha\delta}(X;p_X,R)$ is contained in the image of i_* below:*

$$\begin{array}{ccc} & & L_n^{\delta',\epsilon'}(X;p_X,R)\xrightarrow{j_*}L_n^{\delta'',\epsilon''}(X,W;p_X,R)\\ & & \downarrow\\ L_n^{\{X\},\delta}(Y;p_X,R) & \xrightarrow{i_*} & L_n^{\alpha\delta}(X;p_X,R)\end{array}$$

(3) *Suppose $n\geq 1$, $\delta'\geq\delta$, $\epsilon'\geq\epsilon$, $W\supset Y^{\kappa_n\delta'}$, $\delta''\geq\kappa_n\delta'$, and $\epsilon''\geq\kappa_n\epsilon'$ so that the following two maps are defined:*

$$L_n^{\delta,\epsilon}(X;p_X,R)\xrightarrow{j_*}L_n^{\delta',\epsilon'}(X,Y;p_X,R)\xrightarrow{\partial}L_{n-1}^{\{X\},\delta'',\epsilon''}(W;p_X,R).$$

Then ∂j_* is zero.

(4) *Suppose $n\geq 1$, $W\supset Y^{\kappa_n\delta'}$, $\delta''\geq\kappa_n\delta'$, and $\epsilon''\geq\kappa_n\epsilon'$ so that the map*

$$\partial\colon L_n^{\delta',\epsilon'}(X,Y;p_X,R)\to L_{n-1}^{\{X\},\delta'',\epsilon''}(W;p_X,R)$$

is defined. If $\delta\geq\lambda_n\delta''$ and $Y'\supset W^{\lambda_n\delta''}$, then the relax-control image of the kernel of ∂ in $L_n^{\delta}(X,Y';p_X,R)$ is contained in the image of j_* below:

$$\begin{array}{ccc} & & L_n^{\delta',\epsilon'}(X,Y;p_X,R)\xrightarrow{\partial}L_{n-1}^{\{X\},\delta'',\epsilon''}(W;p_X,R)\\ & & \downarrow\\ L_n^{\delta}(X;p_X,R) & \xrightarrow{j_*} & L_n^{\delta}(X,Y';p_X,R)\end{array}$$

(5) *Suppose* $n \geq 1$, $Y' \supset Y^{\kappa_n \delta}$, $\delta' \geq \kappa_n \delta$, $\epsilon' \geq \kappa_n \epsilon$, $\delta'' \geq \alpha\delta'$, *and* $\epsilon'' \geq \alpha\epsilon'$ *so that the following two maps are defined:*

$$L_n^{\delta,\epsilon}(X,Y;p_X,R) \xrightarrow{\partial} L_{n-1}^{\{X\},\delta',\epsilon'}(Y';p_X,R) \xrightarrow{i_*} L_{n-1}^{\delta'',\epsilon''}(X;p_X,R).$$

Then $i_*\partial$ *is zero.*

(6) *Suppose* $n \geq 1$, $\delta'' \geq \alpha\delta'$, *and* $\epsilon'' \geq \alpha\epsilon'$ *so that* $i_* \colon L_{n-1}^{\{X\},\delta',\epsilon'}(Y;p_X,R) \to L_{n-1}^{\delta'',\epsilon''}(X;p_X,R)$ *is defined. If* $\delta \geq \lambda_n \delta''$ *and* $W \supset Y^{\lambda_n \delta''}$, *then the relax-control image of the kernel of* i_* *in* $L_{n-1}^{\{X\},\kappa_n\delta}(W^{\kappa_n\delta};p_X,R)$ *is contained in the image of* ∂ *below:*

$$\begin{tikzcd} & L_{n-1}^{\{X\},\delta',\epsilon'}(Y;p_X,R) \arrow[r, "i_*"] \arrow[d] & L_{n-1}^{\delta'',\epsilon''}(X;p_X,R) \\ L_n^{\delta}(X,W;p_X,R) \arrow[r, "\partial"] & L_{n-1}^{\{X\},\kappa_n\delta}(W^{\kappa_n\delta};p_X,R) & \end{tikzcd}$$

Proof (1) Let $[(C,p),\psi] \in L_n^{\{X\},\delta,\epsilon}(Y;p_X,R)$. There is a 3ϵ chain equivalence $g\colon (C,p)\oplus(\partial\Sigma E) \to (F,1)$ for some n–dimensional free ϵ chain complexes E and F on p_X, and $j_*i_*[(C,p),\psi] \in L_n^{\delta'',\epsilon''}(X,W;p_X,R)$ is represented by $(F, g_\%(\psi\oplus\partial\theta))$, where θ is the trivial quadratic structure on ΣE. Take the sum of

$$(0\colon\ (C,p)\to 0,(0,\psi)), \quad \text{and} \quad (i_{\Sigma E}\colon\ (\partial\Sigma E,1)\to(E^{n-*},1),(0,\partial\theta)).$$

$(0,\psi\oplus\partial\theta)$ is a 2ϵ connected 2ϵ quadratic structure, and it is 2ϵ Poincaré over $X-Y$. Use the chain equivalence g and Corollary 3.9′ to get a 180ϵ connected 24ϵ null-cobordism

$$(F \longrightarrow E^{n-*},\ \ (\chi, g_\%(\psi\oplus\partial\theta)))$$

that is 180ϵ Poincaré over $X - Y^{486\epsilon}$.

(2) Let $[C,\psi] \in L_n^{\delta',\epsilon'}(X;p_X,R)$ and assume $j_*[C,\psi] = 0 \in L_n^{\delta'',\epsilon''}(X,W;p_X,R)$. By Proposition 3.7, there is a $100\delta''$ connected $2\delta''$ null-cobordism

$$(f\colon\ C\to D,(\delta\psi,\psi))$$

that is $100\delta''$ Poincaré over $X - W^{100\delta''}$. Apply the boundary construction to this null-cobordism to get a $4\delta''$ chain map ∂f of $4\delta''$ chain complexes and an n–dimensional $6\delta''$ Poincaré $6\delta''$ quadratic structure on it:

$$\partial f\colon\ \partial C\to\partial D, \qquad \Psi_3 = (\partial\delta\psi,\partial\psi).$$

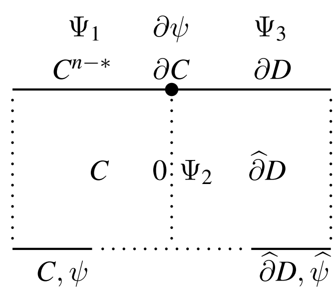
$(\partial C,\partial\psi)$ also appears as the boundaries of

- an n–dimensional $2\epsilon'$ Poincaré $2\epsilon'$ quadratic structure $\Psi_1 = (0, \partial\psi)$ on the ϵ' chain map $i_C\colon\ \partial C \to C^{n-*}$, and
- an n–dimensional ϵ' quadratic structure $\Psi_2 = (0, \partial\psi)$ on the 0 chain map $0\colon\ \partial C \to 0$, which is ϵ' Poincaré because ∂C is $4\epsilon'$ contractible.

The union $\Psi_2 \cup_{\partial C} -\Psi_3$ is a $600\delta''$ Poincaré $7\delta''$ quadratic structure on $0 \cup_{\partial C} \partial D = \mathcal{C}(\partial f)$. By Proposition 3.4(2), there is a $2400\delta''$ chain equivalence between ∂D and an n–dimensional $500\delta''$ chain complex $\widehat{\partial}D$. This chain equivalence, together with the $4\epsilon'$ chain contraction of ∂C, induces a $43200\delta''$ chain equivalence $g\colon\ 0 \cup_{\partial C} \partial D \to \widehat{\partial}D$. Define a $43200\delta''$ Poincaré $3{\cdot}43200\delta''$ quadratic structure $\widehat{\psi}$ on $\widehat{\partial}D$ by $g_{\%}(\Psi_2 \cup_{\partial\psi} -\Psi_3)$. By Proposition 2.9, there is a $43200\delta''$ Poincaré $3\cdot 43200\delta''$ quadratic structure on a $43200\delta''$ chain map

$$(0 \cup_{\partial C} \partial D) \oplus \widehat{\partial}D \longrightarrow \widehat{\partial}D,$$

and, therefore, the right square in the picture below is filled with a cobordism.



The left square can also be filled in with a cobordism. There is a $3\epsilon'$ homotopy equivalence:

$$(C^{n-*} \cup_{\partial C} 0 = \mathcal{C}(i_C), \Psi_1 \cup_{\partial\psi} -\Psi_2) \longrightarrow (C, \psi),$$

and again by Proposition 2.9, this induces a $30\epsilon'$ Poincaré $9\epsilon'$ quadratic structure on a $3\epsilon'$ chain map

$$(C^{n-*} \cup_{\partial C} 0) \oplus C \longrightarrow C.$$

Glue these along the pair $(\partial C \to 0, \Psi_2)$, and we get a chain map

$$(C^{n-*} \cup_{\partial C} \partial D) \oplus C \oplus \widehat{\partial}D \longrightarrow C \oplus \widehat{\partial}D$$

and a $43200000\delta''$ Poincaré $6 \cdot 43200\delta''$ quadratic structure on it. Since ∂C is $4\epsilon'$ contractible and ∂D is $2400\delta''$ chain equivalent to $\widehat{\partial}D$, there is a $43200\delta''$ chain equivalence

$$G\colon\ C^{n-*} \cup_{\partial C} \partial D \longrightarrow E = C^{n-*} \oplus \widehat{\partial}D,$$

and hence, by Corollary 3.9, there is a $30 \cdot 433000000\delta''$ Poincaré $4 \cdot 433000000\delta''$ null-cobordism of $(E, G_{\%}(\Psi_1 \cup_{\partial\psi} -\Psi_3)) \oplus (C, -\psi) \oplus (\widehat{\partial}D, -\widehat{\psi})$. Therefore

$$[C, \psi] + [\widehat{\partial}D, \widehat{\psi}] = [E, G_{\%}(\Psi_1 \cup_{\partial\psi} -\Psi_3)]$$

in $L_n^{13\cdot 10^8\delta''}(X; p_X, R)$.

On the other hand, there is a $600\delta''$ Poincaré null-cobordism of $\Psi_1 \cup_{\partial\psi} -\Psi_3$ on the chain map

$$C^{n-*} \cup_{\partial C} \partial D \longrightarrow \mathcal{C}(f)^{n+1-*}.$$

Using G and Corollary 3.9, we obtain a $30(600+43200+4)\delta''$ Poincaré null-cobordism

$$(E \to \mathcal{C}(f)^{n+1-*}, \quad (\chi, G_{\%}(\Psi_1 \cup_{\partial\psi} -\Psi_3)),$$

and this implies

$$[E, G_{\%}(\Psi_1 \cup_{\partial\psi} -\Psi_3)] = 0 \in L_n^{13\cdot 10^8\delta''}(X; p_X, R)$$

and hence

$$[C, \psi] = -[\widehat{\partial}D, \widehat{\psi}] \in L_n^{13\cdot 10^8\delta''}(X; p_X, R).$$

Since ∂D is $400\epsilon''$ contractible over $X - W^{100\epsilon''}$ and $\widehat{\partial}D$ is $2400\delta''$ chain equivalent to ∂D, $\widehat{\partial}D$ is $5200\delta''$ contractible over $X - W^{2500\delta}$, by Lemma 2.10. By Proposition 4.2, there is a $(6n+15)\cdot 5200\delta''$ chain equivalence h from $(\widehat{\partial}D, 1)$ to an n–dimensional $(3n+12)\cdot 5200\delta''$ projective chain complex (F, p) on $p_{W^{(20800n+75300)\delta''}}$. Suppose $\lambda_n \geq 10^5(4n+50)$. If $\delta \geq \lambda_n\delta''$ and $Y \supset W^{\lambda_n\delta''}$, then $((F,p), h_{\%}(\widehat{\psi}))$ represents an element of $L_n^{\{X\},\delta}(Y; p_Y, R)$ by Proposition 2.9, and its image

$$i_*([(F,p), h_{\%}(\widehat{\psi})]) \in L_n^{\alpha\delta}(X; p_X, R)$$

is represented by $(\widehat{\partial}D, (h^{-1})_{\%}(h_{\%}(\widehat{\psi})) = (h^{-1}h)_{\%}(\widehat{\psi}))$. Since $h^{-1}h$ is 2δ chain homotopic to the identity map,

$$[\widehat{\partial}D, \widehat{\psi}] = [\widehat{\partial}D, (h^{-1}h)_{\%}(\widehat{\psi})] \in L_n^{\alpha\delta}(X; p_X, R).$$

Since $\alpha\delta \geq 13\cdot 10^8\delta''$, we have

$$i_*(-[(F,p), g_{\%}(\widehat{\psi})]) = [C, \psi] \in L_n^{\alpha\delta}(X; p_X, R).$$

(3) Let $[C, \psi] \in L_n^{\delta,\epsilon}(X; p_X, R)$, then ∂C is 4ϵ contractible. Thus $(\partial C, 1)$ is 4ϵ chain equivalent to $(E = 0, q = 0)$.

(4) Let $[C, \psi] \in L_n^{\delta',\epsilon'}(X, Y; p_X, R)$ such that $\partial[C, \psi] = 0$ in $L_{n-1}^{\{X\},\delta'',\epsilon''}(W; p_X, R)$. Let $\beta\colon (\partial C, 1) \to (E, q)$ be a $(200n+300)\epsilon'$ chain equivalence to an $(n-1)$–dimensional $(100n+300)\epsilon'$ projective chain complex on p_W posited in the definition of ∂. By

assumption, $[(E,q),\beta_{\%}\partial\psi]=0$ in $L_{n-1}^{\{X\},\delta'',\epsilon''}(W;p_X,R)$. By Proposition 4.4, there is an $(n-1)$–dimensional $100\delta''$ Poincaré $2\delta''$ null-cobordism on p_W

$$(f'\colon (E,q)\longrightarrow(D,p),\quad (\delta\psi',\beta_{\%}\partial\psi))$$

such that $[D,p]=0$ in $\widetilde{K}_0^{n,9\delta''}(X;p_X,R)$. By Lemma 2.10$'$, $(\delta\psi',\partial\psi)$ is a $125\delta''$ Poincaré $2\delta''$ quadratic structure on the $3\delta''$ chain map

$$f=f'\circ\beta\colon (\partial C,1)\longrightarrow(D,p).$$

On the other hand, $(0,\partial\psi)$ is a $2\epsilon'$ Poincaré $2\epsilon'$ quadratic structure on the ϵ' chain map

$$i_C\colon (\partial C,1)\longrightarrow(C^{n-*},1).$$

Gluing these together, we obtain a $12500\delta''$ Poincaré $4\delta''$ quadratic structure

$$\psi'=(0,\partial\psi)\cup_{\partial\psi}-(\delta\psi',\partial\psi)$$

on $(C',p')=(C^{n-*},1)\cup_{(\partial C,1)}(D,p)$. Since $n\geq 1$, (D,p) is $540\delta''$ chain equivalent to an n–dimensional free $270\delta''$ chain complex $(F,1)$ on p_X by Proposition 4.1.

Assume $n\geq 2$. In this case ∂C is $12\epsilon'$ chain equivalent to an $(n-1)$–dimensional $4\epsilon'$ chain complex $\widehat{\partial}C$, by Proposition 3.4. Using these chain equivalences and Proposition 2.6, we can construct a $6528\delta''$ chain equivalence

$$\gamma\colon (C',p')\longrightarrow(C''=C^{n-*}\cup_{\widehat{\partial}C}F,1).$$

If $\delta\geq 9\cdot 10^5\delta''$, then $(C'',\psi''=\gamma_{\%}\psi')$ determines an element of $L_n^{\delta}(X;p_X,R)$. Suppose $\delta\geq 4\cdot 10^6\delta''$ and $Y'\supset W^{12\cdot 10^6\delta''}$. We shall show that its image by j_* is equal to the relax-control image of $[C,\psi]$ in $L_n^{\delta}(X,Y';p_X,R)$.

Since (D,p) lies over W, it is 0 contractible over $X-W$. Therefore, by Proposition 2.6, the chain map $G\colon (C',p')\to(C^{n-*}\cup_{\partial C}0,1)$ defined by

$$G=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\colon (C^{n-r},1)\oplus(\partial C_{r-1},1)\oplus(D_r,p)\longrightarrow(C^{n-r},1)\oplus(\partial C_{r-1},1)$$

is a $18\delta''$ chain equivalence over $X-W^{6\delta''}$. Furthermore one can easily check that G is 0 connected and that $G_{\%}(\psi')=(0,\partial\psi)\cup_{\partial\psi}-(0,\partial\psi)$. Compose G with a $3\epsilon'$ homotopy equivalence

$$((C^{n-*}\cup_{\partial C}0,1),(0,\partial\psi)\cup_{\partial\psi}-(0,\partial\psi))\longrightarrow((C,1),\psi)$$

to get a $3\epsilon'$ connected $19\delta''$ chain equivalence over $X-W^{7\delta''}$:

$$H\colon (C',p')\longrightarrow(C,1);\qquad H_{\%}(\psi')=\psi.$$

By Proposition 2.9, there is a $3\cdot 12500\delta''$ connected $3\cdot 19\delta''$ quadratic structure $(0,\psi'\oplus-\psi)$ on a chain map $(C',p')\oplus(C,1)\to(C,1)$ that is $125000\delta''$ Poincaré over

$X - W^{3750007\delta''}$. Now use the $6528\delta''$ chain equivalence $\gamma\colon (C',p') \to (C'',1)$ and Corollary 3.9$'$ to this cobordism to obtain an $(n{+}1)$–dimensional δ cobordism between (C'',ψ'') and (C,ψ) that is δ connected and δ Poincaré over $X - Y'$.

In the $n = 1$ case, use the non-positive chain complex obtained from ∂C by applying the folding argument from top instead of $\widehat{\partial}C$. See the proof of Proposition 3.4(1).

(5) Let $[C,\psi] \in L_n^{\delta,\epsilon}(X,Y;p_X,R)$ and let $\beta\colon (\partial C,1) \to (E,q)$ be as in the definition of ∂; $\partial[C,\psi]$ is given by $[(E,q),\beta_\%\partial\psi]$. There exist $(n{-}1)$–dimensional free ϵ' chain complexes E', F on p_X and a $3\epsilon'$ chain equivalence

$$g\colon (E,q) \oplus (\partial\Sigma E',1) \longrightarrow (F,1)$$

with $\Sigma E'$ given the trivial quadratic structure θ, and $i_*[(E,q),\beta_\%\partial\psi]$ is represented by $(F, g_\%((\beta_\%\partial\psi) \oplus \partial\theta))$. We construct a δ'' Poincaré null-cobordism of this.

Take the direct sum of the algebraic Poincaré thickenings of (C,ψ) and $(\Sigma E',\theta)$ to get an ϵ' Poincaré pair

$$(\partial C \oplus \partial\Sigma E' \longrightarrow C^{n-*} \oplus E'^{n-1-*}, \quad (0 \oplus 0, \partial\psi \oplus \partial\theta)).$$

Apply the $4\epsilon'$ chain equivalence

$$\partial C \oplus \partial\Sigma E' = (\partial C,1) \oplus (\partial\Sigma E',1) \xrightarrow{\beta\oplus 1} (E,q) \oplus (\partial\Sigma E',1) \xrightarrow{g} (F,1) = F,$$

to this pair, to obtain an ϵ'' Poincaré null-cobordism of $(F, g_\%((\beta_\%\partial\psi) \oplus \partial\theta))$. (If $n \geq 2$, then we may assume $E' = 0$, and the proof can be much simplified.)

(6) Take an element $[(C,p),\psi] \in L_{n-1}^{\{X\},\delta',\epsilon'}(Y;p_X,R)$ and assume $i_*[(C,p),\psi] = 0$ in $L_{n-1}^{\delta'',\epsilon''}(X;p_X,R)$. By definition of i_*, there exist $(n{-}1)$–dimensional free ϵ' chain complexes E, F on p_X and a $3\epsilon'$ chain equivalence $g\colon (C,p) \oplus (\partial\Sigma E,1) \to (F,1)$ such that $i_*[(C,p),\psi] = [F, g_\%(\psi \oplus \partial\theta)]$. Here θ is the trivial quadratic structure on ΣE. By Proposition 3.7, there is an n–dimensional $100\delta''$ Poincaré $2\delta''$ null-cobordism on p_X of $(F, g_\%(\psi \oplus \partial\theta))$:

$$(f\colon F \longrightarrow D, \quad (\delta\psi, g_\%(\psi \oplus \partial\theta))).$$

By Lemma 2.10$'$, we obtain a $127\delta''$ Poincaré $3\delta''$ null-cobordism:

$$(f \circ g\colon (C,p) \oplus (\partial\Sigma E,1) \longrightarrow (D,1), \quad (\delta\psi, \psi \oplus \partial\theta)).$$

Take the union of this with the 0 connected ϵ' projective quadratic pair

$$((C,p) \longrightarrow 0, \quad (0,-\psi)),$$

which is 0 Poincaré over $X - Y$, and the $3\epsilon'$ Poincaré $3\epsilon'$ quadratic pair

$$(i_{\Sigma E}\colon (\partial\Sigma E,1) \longrightarrow (E^{n-*},1), \quad (0,-\partial\theta))$$

to get a $6\delta''$ projective quadratic complex $((\widehat{C},\widehat{r}),\widehat{\psi})$ which is $12700\delta''$ Poincaré over $X - Y^{12700\delta''}$ and is $12700\delta''$ connected.

The $3\epsilon'$ chain equivalence g induces a $48\delta''$ chain equivalence $\tilde{g}\colon (\widehat{C},\widehat{r}) \to (\tilde{C},1)$ to an n–dimensional free chain complex $(\tilde{C},1) = (D,1) \cup_{(F,1)} (E^{n-*},1)$, and the $18\delta''$ quadratic structure $\tilde{\psi} = \tilde{g}_{\%}(\widehat{\psi})$ is $2\cdot 10^5\delta''$ Poincaré over $X - Y^{4\cdot 10^5\delta''}$ and is $2\cdot 10^5\delta''$ connected. Suppose $W \supset Y^{10^6\delta''}$ and $\delta \geq 10^6\delta''$. Then $(\tilde{C},\tilde{\psi})$ defines an element in $L^{\delta}_n(X,W;p_X,R)$.

We shall show that $\partial[\tilde{C},\tilde{\psi}] = [(C,p),\psi]$ in $L^{\{X\},\kappa_n\delta}_{n-1}(W^{\kappa_n\delta};p_X,R)$. By the definition of ∂, there is a $(200n+300)\delta$ chain equivalence $\beta\colon (\partial\tilde{C},1) \to (\tilde{E},\tilde{q})$ to an $(n-1)$–dimensional $(100n+300)\delta$ projective chain complex on $p_{W^{(150n+300)\delta}}$, and $\partial[\tilde{C},\tilde{\psi}]$ is represented by $((\tilde{E},\tilde{q}),\beta_{\%}\partial\tilde{\psi})$. We construct a cobordism between $((C,p),\psi)$ and $((\tilde{E},\tilde{q}),\beta_{\%}\partial\tilde{\psi})$.

By Proposition 2.9$'$, $\tilde{g}$ induces an $(n+1)$–dimensional $3\cdot 48\delta''$ cobordism:

$$\left(\begin{pmatrix}\tilde{g} & 1\end{pmatrix} : (\widehat{C},\widehat{r}) \oplus (\tilde{C},1) \longrightarrow (\tilde{C},1), \quad \tilde{\Psi} = (0,\widehat{\psi} \oplus -\tilde{\psi})\right).$$

Let us apply the boundary construction to this to get a $6\cdot 48\delta''$ chain map

$$\partial\begin{pmatrix}\tilde{g} & 1\end{pmatrix} : \partial(\widehat{C},\widehat{r}) \oplus \partial(\tilde{C},1) \longrightarrow (G,q)$$

and a $9\cdot 48\delta''$ Poincaré $6\cdot 48\delta''$ quadratic structure $(\chi,\partial\widehat{\psi} \oplus -\partial\tilde{\psi})$ on it. We modify this to get the desired cobordism.

Firstly, note that $((\widehat{C},\widehat{r}),\widehat{\psi})$ is the algebraic Thom complex of a $12700\delta''$ Poincaré $6\delta''$ quadratic pair with boundary equal to $((C,p),\psi)$. Therefore there is a $11\cdot 12700\delta''$ chain equivalence $\gamma\colon\ \partial(\widehat{C},\widehat{r}) \to (C,p)$ such that $\gamma_{\%}(\partial\widehat{\psi}) = \psi$.

Secondly, there is a chain equivalence $\beta\colon\ (\partial\tilde{C},1) \to (\tilde{E},\tilde{q})$ as noted above.

Thirdly, recall that (G,q) is equal $\Omega\mathcal{C}(\mathcal{D}_{\tilde{\Psi}})$ and $\partial(\tilde{C},1)$ is equal to $\Omega\mathcal{C}(\mathcal{D}_{\tilde{\psi}})$, and note that there is a $96\delta''$ chain equivalence

$$\mathcal{C}(\begin{pmatrix}\tilde{g} & 1\end{pmatrix})^{n+1-*} \xrightarrow{\begin{pmatrix}0 & 1 & -\tilde{g}^*\end{pmatrix}} (\widehat{C},\widehat{r}) \xrightarrow{(\tilde{g}^{-1})^*} (\tilde{C},1)$$

and that it induces a 6δ chain equivalence from (G,q) to $\partial(\tilde{C},1)$. Compose this with β to get a $(200n+306)\delta$ chain equivalence $\beta'\colon\ (G,q) \to (\tilde{E},\tilde{q})$.

Now, by Corollary 3.9$'$, one can conclude that the chain equivalences γ, β, β' induce an n–dimensional $\kappa_n\delta$ Poincaré cobordism on $p_{W^{\kappa_n\delta}}$:

$$((C,p) \oplus (\tilde{E},\tilde{q}) \longrightarrow (\tilde{E},\tilde{q}), \quad (\chi,\psi \oplus -\beta_{\%}(\partial\tilde{\psi}))).$$

Since $[C,p] = 0$ in $\widetilde{K}^{n,\epsilon'}_0(X;p_X,R)$ and $[\tilde{E},\tilde{q}] = [\partial\tilde{C},1] = 0$ in $\widetilde{K}^{n,\kappa_n\delta}_0(X;p_X,R)$, this implies that $[(C,p),\psi] = \partial[\tilde{C},\tilde{\psi}]$ in $L^{\{X\},\kappa_n\delta}_{n-1}(W^{\kappa_n\delta};p_X,R)$. □

6 Excision

In this section we study the excision property of epsilon-controlled L–theory. Suppose that X is the union of two closed subsets A and B with intersection $M = A \cap B$. There is an inclusion-induced homomorphism

$$i_* \colon\ L_n^{\delta,\epsilon}(A, M; p_A, R) \to L_n^{\delta,\epsilon}(X, B; p_X, R).$$

For $n \geq 1$, we construct its stable inverse

$$\mathrm{exc} \colon\ L_n^{\delta,\epsilon}(X, B; p_X, R) \to L_n^{\delta,\epsilon}(A, A \cap M^{(n+5)4\delta}; p_A, R).$$

First we define geometric subcomplexes and quotient complexes of free chain complexes. Let C be a free chain complex on p_X. When each C_r is the direct sum $C_r = C'_r \oplus C''_r$ of two geometric submodules and d_C is of the form

$$\begin{pmatrix} d_{C'} & * \\ 0 & d_{C''} \end{pmatrix} \colon\ C'_r \oplus C''_r \longrightarrow C'_{r-1} \oplus C''_{r-1},$$

C' is said to be a *geometric subcomplex* of C, and C'' (together with $d_{C''}$) is said to be the *quotient* of C by C' and is denoted C/C'. If C is a free ϵ chain complex, then any geometric subcomplex C' and the quotient C/C' are both free ϵ chain complexes. The obvious projection map $p \colon\ C \to C/C'$ is 0 connected.

Next suppose we are given an n–dimensional ϵ quadratic complex (C, ψ) on p_X and C' is a geometric subcomplex of C. The projection $p \colon\ C \to C/C'$ induces an n–dimensional ϵ quadratic complex $(C/C', p_{\%}\psi)$ and there is an ϵ cobordism between (C, ψ) and $(C/C', p_{\%}\psi)$. For a morphism $g \colon\ G \to H$ between geometric modules and geometric submodules $G' \subset G$ and $H' \subset H$, we write $g(G') \subset H'$ when every path with non-zero coefficient in g starting from a generator of G' ends at a generator of H'.

Proposition 6.1 *Let (C, ψ), C', and p be as above, and suppose (C, ψ) is ϵ connected. If $\mathcal{D}_\psi(C'^n) \subset C'_0$, then $(C/C', p_{\%}\psi)$ and the cobordism between (C, ψ) and $(C/C', p_{\%}\psi)$ induced by p are both ϵ connected.*

Proof Let us write $C'' = C/C'$. By assumption, the morphism $d_{\mathcal{C}(\mathcal{D}_\psi)} \colon\ \mathcal{C}(\mathcal{D}_\psi)_1 \to \mathcal{C}(\mathcal{D}_\psi)_0$ can be expressed by a matrix of the form

$$\begin{pmatrix} d_{C'} & * & * & * \\ 0 & d_{C''} & 0 & \mathcal{D}_{p_{\%}\psi} \end{pmatrix} \colon\ C'_1 \oplus C''_1 \oplus C'^n \oplus C''^n \to C'_0 \oplus C''_0.$$

Let $h\colon\ C_0 = \mathcal{C}(\mathcal{D}_\psi)_0 \to \mathcal{C}(\mathcal{D}_\psi)_1$ be a 4ϵ morphism such that $d_{\mathcal{C}(\mathcal{D}_\psi)}h \sim_{8\epsilon} 1_{C_0}$, and define 4ϵ morphisms $h_1\colon\ C''_0 \to C''_1$ and $h_2\colon\ C''_0 \to C''^n$ by

$$h = \begin{pmatrix} * & * \\ * & h_1 \\ * & * \\ * & h_2 \end{pmatrix} :\ C'_0 \oplus C''_0 \to C'_1 \oplus C''_1 \oplus C'^n \oplus C''^n.$$

Then we get a homotopy

$$d_{C''}h_1 + \mathcal{D}_{p_\%\psi}h_2 \ \sim_{8\epsilon}\ 1_{C''_0}.$$

Therefore

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} :\ C''_0 \longrightarrow C''_1 \oplus C''^n$$

gives a desired splitting of the boundary morphism $\mathcal{C}(\mathcal{D}_{p_\%\psi})_1 \to \mathcal{C}(\mathcal{D}_{p_\%\psi})_0$. Therefore $(C'', p_\%\psi)$ is ϵ connected. Now the ϵ connectivity of cobordism induced by p follows from Lemma 3.3. □

Example 6.2 Let (C,ψ) be an n–dimensional ϵ quadratic complex on p_X and Y be a subset of X. Fix $\delta(>0)$ and $l(\geq 0)$, and define a geometric submodule C'_r of C_r to be the restriction $C_r(Y^{(n+l-r)\delta})$ of C_r to $Y^{(n+l-r)\delta}$. If $\delta \geq \epsilon$, $\{C'_r\}$ is a geometric subcomplex of C, and we can form the quotient C/C' of C by C' and the natural projection $p\colon\ C \to C/C'$. $(C/C')_r$ is equal to $C_r(X - Y^{(n+l-r)\delta})$. Suppose further that (C,ψ) is ϵ connected, $\delta \geq 4\epsilon$, and $n \geq 1$; then $\mathcal{D}_\psi(C'^n) \subset C'_0$ holds, and $(C/C', p_\%\psi)$ and the cobordism between (C,ψ) and $(C/C', p_\%\psi)$ induced by p are both ϵ connected.

Next we consider pairs. Suppose $(f\colon\ C \to D, (\delta\psi, \psi))$ is an $(n{+}1)$–dimensional ϵ quadratic pair on p_X and C', D' are geometric subcomplexes of C, D, respectively such that $f(C'_r) \subset (D'_r)$ for every r. Define an ϵ chain map $\bar f\colon\ C/C' \to D/D'$ by

$$f = \begin{pmatrix} * & * \\ 0 & \bar f \end{pmatrix} :\ C'_r \oplus (C/C')_r \longrightarrow D'_r \oplus (D/D')_r,$$

then the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ C/C' & \xrightarrow[\bar f]{} & D/D' \end{array}$$

commutes strictly, where p and q are the natural projections, and

$$(\bar f\colon\ C/C' \to D/D', (q_\%\delta\psi, p_\%\psi))$$

is an $(n{+}1)$–dimensional ϵ quadratic pair.

Proposition 6.3 *If $(f,(\delta\psi,\psi))$ is ϵ connected, $\mathcal{D}_\psi(C'^n)\subset C'_0$, and $\mathcal{D}_{\delta\psi}(D'^{n+1})\subset D'_0$, then $(\bar f,(q_\%\delta\psi,p_\%\psi))$ is ϵ connected.*

Proof We check the ϵ connectivity of the duality map $\mathcal{D}_{(q_\%\delta\psi,p_\%\psi)}$. Let us use the notation $C''=C/C'$ and $D''=D/D'$. The boundary morphism

$$d_{\mathcal{C}(\mathcal{D}_{(\delta\psi,\psi)})}\colon\ \mathcal{C}(\mathcal{D}_{(\delta\psi,\psi)})_1\longrightarrow\mathcal{C}(\mathcal{D}_{(\delta\psi,\psi)})_0$$

can be expressed by a matrix of the form

$$\begin{pmatrix} * & * & * & * & * & * \\ 0 & d_{D''} & 0 & \mathcal{D}_{q_\%\delta\psi} & 0 & \bar f\mathcal{D}_{p_\%\psi}\end{pmatrix}:\ D'_1\oplus D''_1\oplus D'^{n+1}\oplus D''^{n+1}\oplus C'^n\oplus C''^n\longrightarrow D'_0\oplus D''_0.$$

The desired ϵ connectivity follows from this as in Proposition 6.1. □

Proposition 6.4 *Let Y be a subset of X, and let $[(C,d),\psi]$ and $[(\widehat C,\widehat d),\widehat\psi]$ be elements of $L_n^{\delta,\epsilon}(X,Y;p_X,R)$ $(n\geq 1)$. If*

(1) $C_r(X-Y)=\widehat C_r(X-Y)$,

(2) $d_r|X-Y^{4\epsilon}=\widehat d_r|X-Y^{4\epsilon}$, *and*

(3) $\psi_s|X-Y^{4\epsilon}=\widehat\psi_s|X-Y^{4\epsilon}$

for every r and s (≥ 0), then $[(C,d),\psi]=[(\widehat C,\widehat d),\widehat\psi]$ in $L_n^{\delta,\epsilon}(X,Y^{(n+3)4\epsilon};p_X,R)$.

Proof Define a geometric subcomplex C' of C by $C'_r=C_r(Y^{(n+1-r)4\epsilon})$, and let $p\colon C\to C/C'$ be the projection. Then $(C/C',p_\%\psi)$ is an ϵ connected ϵ quadratic complex by Proposition 6.1. The boundary maps for $\mathcal{C}(\mathcal{D}_\psi)$ have radius 4ϵ and are of the form

$$\begin{pmatrix} d_{C'} & * & * & * \\ 0 & d_{C''} & * & (-)^{r-1}\mathcal{D}_{p_\%\psi} \\ 0 & 0 & * & * \\ 0 & 0 & * & (-)^{r-1}d^*_{C''}\end{pmatrix}:\ C'_r\oplus C''_r\oplus C'^{n+1-r}\oplus C''^{n+1-r}\longrightarrow C'_{r-1}\oplus C''_{r-1}\oplus C'^{n+2-r}\oplus C''^{n+2-r}.$$

Therefore $\mathcal{C}(\mathcal{D}_{p_\%\psi})$ and $\mathcal{C}(\mathcal{D}_\psi)$ are exactly the same over $X-Y^{(n+2)4\epsilon}$, and $\mathcal{C}(\mathcal{D}_{p_\%\psi})$ is 4ϵ contractible over $X-Y^{(n+3)4\epsilon}$, ie $p_\%\psi$ is ϵ Poincaré over $X-Y^{(n+3)4\epsilon}$. In fact, if Γ is a 4ϵ chain contraction over $X-Y$ of $\mathcal{C}(\mathcal{D}_\psi)$, then $\Gamma|X-Y^{(n+2)4\epsilon}$ gives a 4ϵ chain contraction over $X-Y^{(n+3)4\epsilon}$ of $\mathcal{C}(\mathcal{D}_{p_\%\psi})$. Thus $(C/C',p_\%\psi)$ determines an element of $L_n^{\delta,\epsilon}(X,Y^{(n+3)4\epsilon};p_X,R)$.

By Lemma 3.3, the cobordism between (C,ψ) and $(C/C',p_{\%}\psi)$ induced by p is an ϵ connected ϵ quadratic pair. Since this cobordism is exactly the same over $X-Y^{(n+2)4\epsilon}$ as the trivial cobordism from (C,ψ) to itself, it is ϵ Poincaré over $X-Y^{(n+3)4\epsilon}$. Therefore,

$$[C,\psi] \;=\; [C/C',p_{\%}\psi] \;\in\; L_n^{\delta,\epsilon}(X,Y^{(n+3)4\epsilon};p_X,R).$$

The same construction for $(\widehat{C},\widehat{\psi})$ yields the same element as this, and we can conclude that

$$[C,\psi] \;=\; [\widehat{C},\widehat{\psi}] \;\in\; L_n^{\delta,\epsilon}(X,Y^{(n+3)4\epsilon};p_X,R). \qquad \square$$

Now suppose X is the union of two closed subsets A, B with intersection $N=A\cap B$.

Lemma 6.5 *Let G, H be geometric modules on p_X, and $f\colon G\to H$ be a morphism of radius δ. Then, for any $\gamma\geq 0$,*

$$f(G(B\cup N^{\gamma})) \;\subset\; H(B\cup N^{\max\{\gamma+\delta,2\delta\}}).$$

Proof This can be deduced from the following two claims:

(1) $f(G(N^{\gamma})) \;\subset\; H(N^{\gamma+\delta})$,

(2) $f(G(B)) \;\subset\; H(B\cup N^{2\delta})$.

The first claim is obvious. To prove the second claim, take a generator of $G(B)$ and a path c starting from a with non-zero coefficient in f. By its continuity, the path $p_X\circ c$ in X either stays inside of B or passes through a point in N, and hence its image is contained in $B\cup N^{2\delta}$. This proves the second claim. $\square$

Now let us define the excision map:

$$\mathrm{exc}\colon\; L_n^{\delta,\epsilon}(X,B;p_X,R)\to L_n^{\delta,\epsilon}(A,A\cap N^{(n+5)4\delta};p_A,R),$$

Take an element $[C,\psi]\in L_n^{\delta,\epsilon}(X,B;p_X,R)$. Define a geometric subcomplex C' of C by

$$C'_r \;=\; C_r(B\cup N^{(n+2-r)4\epsilon}),$$

and let $p\colon\; C\to C/C'$ denote the projection. Then, by Lemma 6.5 and Proposition 6.1, $(C/C',p_{\%}\psi)$ is an ϵ connected ϵ quadratic complex on p_A and is ϵ Poincaré over $A-N^{(n+4)4\epsilon}$. We define $\mathrm{exc}([C,\psi])$ to be the element

$$[C/C',p_{\%}\psi] \;\in\; L_n^{\delta,\epsilon}(A,A\cap N^{(n+5)4\epsilon};p_A,R).$$

The excision map is well-defined. Suppose

$$[C,\psi] \;=\; [\widehat{C},\widehat{\psi}] \;\in\; L_n^{\delta,\epsilon}(X,B;p_X,R).$$

Without loss of generality we may assume that there is a δ connected δ cobordism

$$(f\colon\ C \oplus \widehat{C} \to D, (\delta\psi, \psi \oplus -\widehat{\psi}))$$

between (C,ψ) and $(\widehat{C},\widehat{\psi})$ that is δ Poincaré over $X - B$. Let us now construct $(C/C', p_{\%}\psi)$ and $(\widehat{C}/\widehat{C}', \widehat{p}_{\%}\widehat{\psi})$ as above, define a geometric subcomplex D' of D by

$$D'_r \;=\; D_r(B \cup N^{(n+3-r)4\delta}),$$

and let $q\colon\ D \to D/D'$ denote the projection. By Lemma 6.5 and Proposition 6.3, we obtain an δ connected δ cobordism

$$(\bar{f}\colon\ C/C' \oplus \widehat{C}/\widehat{C}' \to D/D', (q_{\%}\delta\psi, p_{\%}\psi \oplus -\widehat{p}_{\%}\widehat{\psi}))$$

which is δ Poincaré over $A - B \cup N^{(n+5)4\delta}$. Therefore exc is well-defined.

By using Proposition 6.4, we can check that the homomorphisms i_* and exc are stable inverses; ie the following diagram commutes:

$$\begin{tikzcd}
L_n^{\delta,\epsilon}(A,N;p_A,R) \arrow[r, "i_*"] \arrow[d] & L_n^{\delta,\epsilon}(X,B;p_X,R) \arrow[d, equal] \\
L_n^{\delta,\epsilon}(A,A\cap N^{(n+5)4\delta};p_A,R) \arrow[d, equal] & L_n^{\delta,\epsilon}(X,B;p_X,R) \arrow[l, "\mathrm{exc}"'] \arrow[d] \\
L_n^{\delta,\epsilon}(A,A\cap N^{(n+5)4\delta};p_A,R) \arrow[r, "i_*"'] & L_n^{\delta,\epsilon}(X,B\cup N^{(n+5)4\delta};p_X,R)
\end{tikzcd}$$

where the vertical maps are the homomorphisms induced by inclusion maps.

7 Mayer–Vietoris type sequence

We continue to assume that X is the union of two closed subsets A, B with intersection $N = A \cap B$, and will present a Mayer–Vietoris type stably exact sequence.

Replace κ_n by $\kappa_n + 4(n+5)$, and suppose $\delta \geq \epsilon > 0$. Let W be a subset of X containing $N^{\kappa_n\delta}$ and assume $\delta' \geq \kappa_n\delta$, $\epsilon' \geq \kappa_n\epsilon$. Then a homomorphism

$$\bar{\partial}\colon\ L_n^{\delta,\epsilon}(X;p_X,R) \longrightarrow L_{n-1}^{\{A\cup W\},\delta',\epsilon'}(W;p_{A\cup W},R)$$

is obtained by composing the following maps:

$$\begin{aligned}
L_n^{\delta,\epsilon}(X;p_X,R) \longrightarrow L_n^{\delta,\epsilon}(X,B;p_X,R) &\xrightarrow{\mathrm{exc}} L_n^{\delta,\epsilon}(A,A\cap N^{(n+5)4\delta};p_A,R) \\
&\xrightarrow{\partial} L_{n-1}^{\{A\},\delta',\epsilon'}(A\cap W;p_A,R) \longrightarrow L_{n-1}^{\{A\cup W\},\delta',\epsilon'}(W;p_{A\cup W},R)
\end{aligned}$$

If $[C,\psi]\in L_n^{\delta,\epsilon}(X;p_X,R)$, then its image $\overline{\partial}[C,\psi]$ is represented by $((E,q),\psi')$ which is homotopy equivalent to the boundary $\widehat{\partial}(C/C',p_{\%}\psi)$, where $C'\subset C$ and $p\colon\ C\to C/C'$ are as in the definition of exc. This is exactly the projective quadratic Poincaré complex $(Q,\overline{\psi})$ which appears in the Splitting Lemma:

Lemma 7.1 (Pedersen–Yamasaki [5]) *For any integer $n\geq 2$, there exists a positive number $\kappa_n\geq 1$ such that the following holds: Suppose $p_X\colon\ M\to X$ is a map to a metric space X, X is the union of two closed subsets A and B with intersection $N=A\cap B$, and R is a ring with involution. Let ϵ be any positive number, and set $\epsilon'=\kappa_n\epsilon$, $N'=N^{\epsilon'}$, $A'=A\cup N'$, and $B'=B\cup N'$. Then for any n–dimensional quadratic Poincaré R–module complex $c=(C,\psi)$ on p_X of radius ϵ, there exist a Poincaré cobordism of radius ϵ' from c to the union $c'\cup c''$ of an n–dimensional quadratic Poincaré pair $c'=(f'\colon\ Q\to C',(\delta\overline{\psi}',-\overline{\psi}))$ on $p_{A'}$ of radius ϵ' and an n–dimensional quadratic Poincaré pair $c''=(f''\colon\ Q\to C'',(\delta\overline{\psi}'',\overline{\psi}))$ on $p_{B'}$ of radius ϵ' along an $(n{-}1)$–dimensional quadratic Poincaré projective R–module complex $(Q,\overline{\psi})$ on $p_{N'}$, where Q is ϵ' chain equivalent to an $(n{-}1)$–dimensional free chain complex on $p_{A'}$ and also to an $(n{-}1)$–dimensional free chain complex on $p_{B'}$.*

From this and its relative version, we obtain the following:

Proposition 7.2 *If $n\geq 2$, the map $\overline{\partial}$ factors through a homomorphism*

$$\partial\colon\ L_n^{\delta,\epsilon}(X;p_X,R)\longrightarrow L_{n-1}^{\mathcal{F},\delta',\epsilon'}(W;p_X,R),$$

where $\mathcal{F}=\{A\cup W,A\cup W\}$. Moreover the image $\partial[C,\psi]$ is given by $[Q,\overline{\psi}]$ which appears in any splitting (up to cobordism) of (C,ψ) according to the closed subsets A, B.

Now we present the Mayer–Vietoris type stably-exact sequence. It is made up of three kinds of maps. The first is the map

$$i_*\colon\ L_n^{\{A,B\},\delta,\epsilon}(N;p_X,R)\to L_n^{\delta',\epsilon'}(A;p_A,R)\oplus L_n^{\delta',\epsilon'}(B;p_B,R)$$

defined by $i_*(x)=(\iota_A(x),-\iota_B(x))$ when $\delta'\geq\alpha\delta$ and $\epsilon'\geq\alpha\epsilon$. The second is the map

$$j_*\colon\ L_n^{\delta,\epsilon}(A;p_A,R)\oplus L_n^{\delta,\epsilon}(B;p_B,R)\to L_n^{\delta',\epsilon'}(X;p_X,R)$$

defined by $j_*(x,y)=j_{A*}(x)+j_{B*}(y)$ when $\delta'\geq\delta$ and $\epsilon'\geq\epsilon$. Here $j_A\colon\ A\to X$ and $j_B\colon\ B\to X$ are inclusion maps. The third is the map ∂ given in Proposition 7.2:

$$\partial\colon\ L_n^{\delta,\epsilon}(X;p_X,R)\longrightarrow L_{n-1}^{\{A\cup W,B\cup W\},\delta',\epsilon'}(W;p_X,R),$$

where $W\supset N^{\kappa_n\delta}$, $\delta'\geq\kappa_n\delta$, and $\epsilon'\geq\kappa_n\epsilon$.

In the rest of this section, we omit the control map and the coefficient ring from notation.

Theorem 7.3 *For any integer $n \geq 2$, there exists a constant $\lambda_n > 1$ which depends only on n such that the following holds true for any control map p_X and two closed subsets A, B of X satisfying $X = A \cup B$:*

(1) *Suppose $\delta' \geq \alpha\delta$, $\epsilon' \geq \alpha\epsilon$, $\delta'' \geq \delta'$, and $\epsilon'' \geq \epsilon'$ so that the following two maps are defined:*

$$L_n^{\{A,B\},\delta,\epsilon}(N) \xrightarrow{i_*} L_n^{\delta',\epsilon'}(A) \oplus L_n^{\delta',\epsilon'}(B) \xrightarrow{j_*} L_n^{\delta'',\epsilon''}(X)$$

Then $j_* i_*$ is zero.

(2) *Suppose $\delta'' \geq \delta'$, $\epsilon'' \geq \epsilon'$ so that $j_*\colon L_n^{\delta',\epsilon'}(A) \oplus L_n^{\delta',\epsilon'}(B) \to L_n^{\delta'',\epsilon''}(X)$ is defined. If $\delta \geq \lambda_n\delta''$ and $W \supset N^{\lambda_n\delta''}$, then the relax-control image of the kernel of j_* in $L_n^{\alpha\delta}(A \cup W) \oplus L_n^{\alpha\delta}(B \cup W)$ is contained in the image of i_* below:*

$$\begin{array}{ccc} & L_n^{\delta',\epsilon'}(A) \oplus L_n^{\delta',\epsilon'}(B) & \xrightarrow{j_*} L_n^{\delta'',\epsilon''}(X) \\ & \downarrow & \\ L_n^{\{A\cup W,B\cup W\},\delta}(W) \xrightarrow{i_*} & L_n^{\alpha\delta}(A \cup W) \oplus L_n^{\alpha\delta}(B \cup W) & \end{array}$$

(3) *Suppose $\delta' \geq \delta$, $\epsilon' \geq \epsilon$, $W \supset N^{\kappa_n\delta'}$, $\delta'' \geq \kappa_n\delta'$, and $\epsilon'' \geq \kappa_n\epsilon'$ so that the following two maps are defined:*

$$L_n^{\delta,\epsilon}(A) \oplus L_n^{\delta,\epsilon}(B) \xrightarrow{j_*} L_n^{\delta',\epsilon'}(X) \xrightarrow{\partial} L_{n-1}^{\{A\cup W,B\cup W\},\delta'',\epsilon''}(W)$$

Then ∂j_* is zero.

(4) *Suppose $W \supset N^{\kappa_n\delta'}$, $\delta'' \geq \kappa_n\delta'$, and $\epsilon'' \geq \kappa_n\epsilon'$ so that the map $\partial\colon L_n^{\delta',\epsilon'}(X) \to L_{n-1}^{\{A\cup W,B\cup W\},\delta'',\epsilon''}(W)$ is defined. If $\delta \geq \lambda_n\delta''$, then the relax-control image of the kernel of ∂ in $L_n^{\delta}(X)$ is contained in the image of j_* below:*

$$\begin{array}{ccc} & L_n^{\delta',\epsilon'}(X) & \xrightarrow{\partial} L_{n-1}^{\{A\cup W,B\cup W\},\delta'',\epsilon''}(W) \\ & \downarrow & \\ L_n^{\delta}(A \cup W) \oplus L_n^{\delta}(B \cup W) \xrightarrow{j_*} & L_n^{\delta}(X) & \end{array}$$

(5) *Suppose $W \supset N^{\kappa_n\delta}$, $\delta' \geq \kappa_n\delta$, $\epsilon' \geq \kappa_n\epsilon$, $\delta'' \geq \alpha\delta'$, and $\epsilon'' \geq \alpha\epsilon'$ so that the following two maps are defined:*

$$L_n^{\delta,\epsilon}(X) \xrightarrow{\partial} L_{n-1}^{\{A\cup W,B\cup W\},\delta',\epsilon'}(W) \xrightarrow{i_*} L_{n-1}^{\delta'',\epsilon''}(A \cup W) \oplus L_{n-1}^{\delta'',\epsilon''}(B \cup W)$$

Then $i_*\partial$ is zero.

(6) *Suppose $\delta'' \geq \alpha\delta'$, and $\epsilon'' \geq \alpha\epsilon'$ so that $i_*\colon\ L_{n-1}^{\{A,B\},\delta',\epsilon'}(N) \to L_{n-1}^{\delta'',\epsilon''}(A) \oplus L_{n-1}^{\delta'',\epsilon''}(B)$ is defined. If $\delta \geq \lambda_n\delta''$, $N' \supset N^{\lambda_n\delta''}$, and $W = (N')^{\kappa_n\delta}$, then the relax-control image of the kernel of i_* in $L_{n-1}^{\{A\cup W, B\cup W\},\delta}(W)$ is contained in the image of ∂ associated with the two closed subsets $A \cup N'$, $B \cup N'$:*

$$\begin{tikzcd} & L_{n-1}^{\{A,B\},\delta',\epsilon'}(N) \arrow[r, "i_*"] \arrow[d] & L_{n-1}^{\delta'',\epsilon''}(A) \oplus L_{n-1}^{\delta'',\epsilon''}(B) \\ L_n^{\delta}(X) \arrow[r, "\partial"] & L_{n-1}^{\{A\cup W, B\cup W\},\delta}(W) & \end{tikzcd}$$

Proof (1) Take an element $x = [Q, \psi] \in L_n^{\{A,B\},\delta,\epsilon}(N)$. The image $i_*(x)$ is a pair $([c_A], -[c_B])$ where c_A and c_B are free quadratic Poincaré complexes on p_A and p_B that are both homotopy equivalent to (Q, ψ), and hence $[c_A] = [c_B] \in L_n^{\delta'',\epsilon''}(X)$. Therefore, $j_*i_*(x) = [c_A] - [c_B] = 0$.

(2) First, temporarily use the constant λ_n posited in the splitting lemma. Take an element $x = ([C_A, \psi_A], [C_B, \psi_B]) \in L_n^{\delta',\epsilon'}(A) \oplus L_n^{\delta',\epsilon'}(B)$ and assume $j_*(x) = 0$. There exists a null-cobordism $(f\colon\ C_A \oplus C_B \to D, (\delta\psi, \psi_A \oplus -\psi_B))$. Its boundary is already split according to A and B, so use the relative splitting to this null-cobordism to get cobordisms of radius $\lambda_n\delta''$:

$$(f_A\colon\ (C_A, 1) \oplus Q \to (D_A, 1), (\delta\psi_A, \psi_A \oplus -\overline{\psi})) \ \text{ on } \ p_{A\cup W},$$
$$(f_B\colon\ (C_B, 1) \oplus Q \to (D_B, 1), (\delta\psi_B, \psi_B \oplus -\overline{\psi})) \ \text{ on } \ p_{B\cup W}.$$

By the Poincaré duality $D_A^{n+1-*} \simeq \mathcal{C}(f_A)$, we have

$$[D_A] - [C_A] - [Q] = [\mathcal{C}(f_A)] = [D_A^{n+1-*}] = 0 \in \widetilde{K}_0^{n+1,4\lambda_n\delta''}(A \cup W),$$

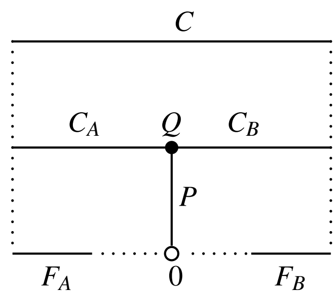
and, hence, we have $[Q] = 0$ in $\widetilde{K}_0^{n,36\lambda_n\delta''}(A \cup W)$. See Ranicki–Yamasaki [14, Section 3]. Similarly $[Q] = 0$ in $\widetilde{K}_0^{n,36\lambda_n\delta''}(B \cup W)$. Thus we obtain an element $[Q, \overline{\psi}]$ of $L_n^{\{A\cup W, B\cup W\},36\lambda_n\delta''}(W)$. Replace λ_n by something bigger (at least $36\lambda_n$) so that its image via i_* in $L_n^{\alpha\delta}(A \cup W) \oplus L_n^{\alpha\delta}(B \cup W)$ is equal to $([C_A, \psi_A], [C_B, \psi_B])$ whenever $\delta \geq \lambda_n\delta''$.

(3) If we start with an element $x = ([C_A, \psi_A], [C_B, \psi_B])$, then $j_*(x)$ is represented by $(C_A, \psi_A) \oplus (C_B, \psi_B)$ which is already split according to A and B. Therefore $\partial j_*(x) = 0$.

(4) Temporarily set the constant λ_n to be the one posited in the splitting lemma. Take an element $[C, \psi]$ in $L_n^{\delta',\epsilon'}(X)$ such that $\partial[C, \psi] = 0$. (C, ψ) splits into two adjacent pairs:

$$a = (f_A\colon\ Q \to (C_A, 1), (\delta\psi_A, -\overline{\psi})) \quad \text{and} \quad b = (f_B\colon\ Q \to (C_B, 1), (\delta\psi_B, \overline{\psi}))$$

such that $[Q,\overline{\psi}] = 0$ in $L_{n-1}^{\{A\cup W, B\cup W\},\delta'',\epsilon''}(W)$. Take a δ'' null-cobordism on p_W $p = (g\colon Q \to P, (\delta\overline{\psi}, \overline{\psi}))$ such that the reduced projective class of P is zero on $p_{A\cup W}$ and also on $p_{B\cup W}$. $C_A \cup_Q P$ is chain equivalent to an n–dimensional free complex F_A on p_A, and $C_B \cup_Q P$ is chain equivalent to an n–dimensional free complex F_B on p_B. Use these to fill in the bottom squares with cobordisms:



Replacing λ_n with something larger if necessary, we obtain free quadratic Poincaré complexes on $p_{A\cup W}$ and $p_{B\cup W}$ whose sum is $\lambda_n\delta''$ cobordant to (C,ψ).

(5) If we start with an element $x = [C,\psi]$ in $L_n^{\delta,\epsilon}(X)$, then $j_*(x)$ is represented by the projective piece $(Q,\overline{\psi})$ obtained by splitting, and the null-cobordisms required to show $i_*\partial(x) = 0$ are easily constructed from the split pieces.

(6) Take an element $[Q,\psi]$ of $L_{n-1}^{\{A,B\},\delta',\epsilon'}(N)$. Then (Q,ψ) is homotopy equivalent to a free quadratic Poincaré complex $((C_A,1),\psi_A)$ on p_A and also to a free quadratic Poincaré complex $((C_B,1),\psi_B)$ on p_B. If $i_*[Q,\psi] = 0$, then these are both null-cobordant; there are quadratic Poincaré pairs

$$\begin{aligned}&(f_A\colon\ C_A \to D_A, (\delta\psi_A, \psi_A)) \quad \text{on } p_A\text{, and}\\ &(f_B\colon\ C_B \to D_B, (\delta\psi_B, \psi_B)) \quad \text{on } p_B.\end{aligned}$$

Use the homotopy equivalence $(C_A,\psi_A) \simeq (C_B,\psi_B)$ to replace the boundary of the latter by (C_A,ψ_B), and glue them together to get an element $[D,\delta\psi]$ of $L_n^{\delta}(X)$ for some $\delta > 0$. Note that $(D,\delta\psi)$ has a splitting into two pairs with the common boundary piece equal to (Q,ψ), so we have $\partial[D,\delta\psi] = [Q,\psi]$ in $L_{n-1}^{\{A\cup W,B\cup W\},\delta}(W)$. □

8 A special case

In this section we treat the case when there are no controlled K–theoretic difficulties.

First assume that X is a finite polyhedron. We fix its triangulation. Under this assumption we can simplify the Mayer–Vietoris type sequence of the previous section at least for sufficiently small ϵ's and δ's. X is equipped with a deformation $\{f_t\colon X \to X\}$ called 'rectification' [5] which deforms sufficiently small neighborhoods of the i–skeleton $X^{(i)}$ into $X^{(i)}$ such that f_t's are uniformly Lipschitz. This can be used to rectify the enlargement of the relevant subsets at the expense of enlargement of ϵ's and δ's. We thank Frank Quinn for showing us his description of *uniformly continuous CW complexes* which are designed for taking care of these situations in a more general setting.

Next let us assume that X is a finite polyhedron and that the control map $p_X\colon M \to X$ is a fibration with path-connected fiber F such that

$$\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = 0$$

for every $k \geq 0$. The condition on the fundamental group is satisfied if $\pi_1(F) \cong \mathbb{Z}^l$ for some $l \geq 0$. If we study the proofs of [14, 8.1 and 8.2] carefully, we obtain the following.

Proposition 8.1 *Let p_X be as above and $n \geq 0$ be an integer. Then there exist numbers $\epsilon_0 > 0$ and $0 < \mu \leq 1$ which depend on X and n such that the relax-control maps*

$$\widetilde{K}_0^{n,\epsilon}(S; p_S, \mathbb{Z}) \longrightarrow \widetilde{K}_0^{n,\epsilon'}(S; p_S, \mathbb{Z})$$
$$\mathrm{Wh}^{n,\epsilon}(S; p_S, \mathbb{Z}) \longrightarrow \mathrm{Wh}^{n,\epsilon'}(S; p_S, \mathbb{Z})$$

are zero maps for any subpolyhedron S, any $\epsilon' \leq \epsilon_0$ and any $\epsilon \leq \mu\epsilon'$.

This means that there is a homomorphism functorial with respect to relaxation of control

$$L_n^{\mathcal{F},\delta,\epsilon}(S; p_X, \mathbb{Z}) \longrightarrow L_n^{\mathcal{F}\cup\{S\},\delta',\epsilon'}(S; p_X, \mathbb{Z})$$

for any family $\mathcal{F}$ of subpolyhedra of X containing S, if $\delta' \leq \epsilon_0$, $\delta \leq \mu\delta'$, and $\epsilon \leq \mu\epsilon'$. Compose this with the homomorphism

$$\iota_S\colon\ L_n^{\mathcal{F}\cup\{S\},\delta',\epsilon'}(S; p_X, \mathbb{Z}) \longrightarrow L_n^{\alpha\delta',\alpha\epsilon'}(S; p_X, \mathbb{Z})$$

to get a homomorphism

$$\iota\colon\ L_n^{\mathcal{F},\delta,\epsilon}(S; p_X, \mathbb{Z}) \longrightarrow L_n^{\alpha\delta',\alpha\epsilon'}(S; p_X, \mathbb{Z}).$$

A stable inverse τ functorial with respect to relaxation of control can be defined by $\tau([C,\psi]) = [(C,1),\psi]$, and we have a commutative diagram:

$$\begin{array}{ccc}
L_n^{\delta,\epsilon}(S;p_X,\mathbb{Z}) & \xrightarrow{\tau} & L_n^{\mathcal{F},\delta,\epsilon}(S;p_X,\mathbb{Z}) \\
\downarrow & & \| \\
L_n^{\alpha\delta',\alpha\epsilon'}(S;p_X,\mathbb{Z}) & \xleftarrow{\iota} & L_n^{\mathcal{F},\delta,\epsilon}(S;p_X,\mathbb{Z}) \\
\| & & \downarrow \\
L_n^{\alpha\delta',\alpha\epsilon'}(S;p_X,\mathbb{Z}) & \xrightarrow[\tau]{} & L_n^{\mathcal{F},\alpha\delta',\alpha\epsilon'}(S;p_X,\mathbb{Z})
\end{array} \quad .$$

Thus the Mayer–Vietoris type sequence is stably exact when we replace the controlled projective L–group terms with appropriate controlled L–groups.

Furthermore, since p_X is a fibration, we have a stability for controlled L–groups:

Proposition 8.2 (Pedersen–Yamasaki [5, Theorem 1]) *Let $n \geq 0$. Suppose Y is a finite polyhedron and $p_Y\colon M \to Y$ is a fibration. Then there exist constants $\delta_0 > 0$ and $K > 1$, which depends on the integer n and Y, such that the relax-control map $L_n^{\delta',\epsilon'}(Y;p_Y,R) \to L_n^{\delta,\epsilon}(Y;p_Y,R)$ is an isomorphism if $\delta_0 \geq \delta \geq K\epsilon$, $\delta_0 \geq \delta' \geq K\epsilon'$, $\delta \geq \delta'$, and $\epsilon \geq \epsilon'$.*

Now let us denote these isomorphic groups $L_n^{\delta,\epsilon}(Y;p_Y,R)$ ($\delta_0 \geq \delta$, $\delta \geq \kappa\epsilon$) by $L_n(Y;p_Y,R)$. When the coefficient ring R is $\mathbb{Z}$, we omit $\mathbb{Z}$ and use the notation $L_n(Y;p_Y)$.

Theorem 8.3 *Let $p_X\colon M \to X$ be a fibration over a finite polyhedron X. Then $L_n(X;p_X,R)$ is 4–periodic: $L_n(X;p_X,R) \cong L_{n+4}(X;p_X,R)$ ($n \geq 0$).*

Proof The proof of the 4–periodicity of $L_n(\mathbb{A})$ of an additive category with involution given in Ranicki [10] adapts well to the controlled setting. □

We have a Mayer–Vietoris exact sequence for L_n with coefficient ring $\mathbb{Z}$.

Theorem 8.4 *Let X be a finite polyhedron and suppose that $p_X\colon M \to X$ is a fibration with path-connected fiber F such that $\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = 0$ for every $k \geq 0$. If X is the union of two subpolyhedra A and B, then there is a long exact sequence*

$$\begin{aligned}
\ldots \xrightarrow{\partial} L_n(A\cap B;p_{A\cap B}) &\xrightarrow{i_*} L_n(A;p_A) \oplus L_n(B;p_B) \xrightarrow{j_*} L_n(X;p_X) \\
&\xrightarrow{\partial} L_{n-1}(A\cap B;p_{A\cap B}) \xrightarrow{i_*} \ldots \xrightarrow{j_*} L_0(X;p_X).
\end{aligned}$$

Proof The exactness at the term $L_2(A;p_A) \oplus L_2(B;p_B)$ and at the terms to the left of it follows immediately from the stably-exact sequence. The exactness at other terms follows from the 4–periodicity. $\square$

Recall that there is a functor $\mathbb{L}(-)$ from spaces to Ω–spectra such that $\pi_n(\mathbb{L}(M)) = L_n(\mathbb{Z}[\pi_1(M)])$ constructed geometrically by Quinn [6], and algebraically by Ranicki [11]. Blockwise application of $\mathbb{L}$ to p_X produces a generalized homology group $H_n(X;\mathbb{L}(p_X))$ (see Quinn [7]). There is a map $A\colon\ H_n(X;\mathbb{L}(p_X)) \to L_n(X;p_X)$ called the assembly map. See Yamasaki [15] for the $\mathbb{L}^{-\infty}$–analogue, involving the lower L–groups of Ranicki [12].

Theorem 8.5 *Let X be a finite polyhedron and suppose that $p_X\colon\ M \to X$ is a fibration with path-connected fiber F such that $\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = 0$ for every $k \geq 0$. Then the assembly map $A\colon\ H_n(X;\mathbb{L}(p_X)) \to L_n(X;p_X)$ is an isomorphism.*

Proof We actually prove the isomorphism $A\colon\ H_n(S;\mathbb{L}(p_S)) \to L_n(S;p_S)$ for all the subpolyhedra S of X by induction on the number of simplices.

When S consists of a single point v, then the both sides are $L_n(\mathbb{Z}[\pi_1(p_X^{-1}(v))])$ and A is the identity map.

Suppose S consists of $k > 1$ simplices and assume by induction that the assembly map is an isomorphism for all subpolyhedra consisting of less number of simplices. Pick a simplex Δ which is not a face of other simplices and let $A = \Delta$ and $B = S - \mathrm{interior}(\Delta)$. Since A contracts to a point v, it can be easily shown that $H_n(A;\mathbb{L}(p_A))$ and $L_n(A;p_A)$ are both $L_n(\mathbb{Z}[\pi_1(p_X^{-1}(v))])$, and the assembly map $A\colon\ H_n(A;\mathbb{L}(p_A)) \to L_n(A;p_A)$ is an isomorphism. By induction hypothesis the assembly maps for B and $A \cap B$ are both isomorphisms. We can conclude that the assembly map for S is an isomorphism by an application of 5–lemma to the ladder made up of the Mayer–Vietoris sequences for $H_*(-)$ and $L_*(-)$. $\square$

Remark If F is simply-connected, then $\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = \mathrm{Wh}(\mathbb{Z}^k) = 0$ for every $k \geq 0$ by the celebrated result of Bass, Heller and Swan. In this case $H_n(X;\mathbb{L}(p_X))$ is isomorphic to the generalized homology group $H_n(X;\mathbb{L})$ where $\mathbb{L}$ is the 4–periodic simply-connected surgery spectrum with $\pi_n(\mathbb{L}) = L_n(\mathbb{Z}[\{1\}])$ and we have an assembly isomorphism

$$A\colon\ H_n(X;\mathbb{L}) \cong L_n(X;p_X).$$

This is the controlled surgery obstruction group which appears in the controlled surgery exact sequence of Pedersen–Quinn–Ranicki [4] (as required for the surgery classification

of exotic homology manifolds in Bryant–Ferry–Mio–Weinberger [1]). There the control map is not assumed to be a fibration. We believe that most of the arguments in this paper work in a more general situation.

As an application of Theorem 8.4, we consider the $\mathbb{Z}$–coefficient controlled L–group of $p_X \times 1\colon\ M \times S^1 \to X \times S^1$.

Corollary 8.6 *Let $n \geq 0$, and let X and $p_X\colon\ M \to X$ be as in Theorem 8.4. Then there is a split short exact sequence*

$$0 \to L_n(X; p_X) \xrightarrow{i_*} L_n(X \times S^1; p_X \times 1) \xrightarrow{B} L_{n-1}(X; p_X) \to 0.$$

Proof Split the circle $S^1 = \partial([-1,1] \times [-1,1]) \subset \mathbb{R}^2$ into two pieces

$$S^1_+ = \{(x,y) \in S^1 | y \geq 0\} \quad \text{and} \quad S^1_- = \{(x,y) \in S^1 | y \leq 0\},$$

with intersection $\{p = (1,0),\ q = (-1,0)\}$. Let ∂ be the connecting homomorphism in the Mayer–Vietoris sequence Theorem 8.4 corresponding to this splitting, and consider the composite

$$\begin{aligned} B\colon\ L_n(X \times S^1; p_X \times 1) &\xrightarrow{\partial} L_{n-1}(X \times \{p\}; p_X \times 1) \oplus L_{n-1}(X \times \{q\}; p_X \times 1) \\ &\xrightarrow{\text{projection}} L_{n-1}(X \times \{p\}; p_X \times 1) \cong L_{n-1}(X; p_X). \end{aligned}$$

Then ∂ can be identified with

$$(B, -B)\colon\ L_n(X \times S^1; p_X \times 1) \longrightarrow L_{n-1}(X; p_X) \oplus L_{n-1}(X; p_X).$$

The map i_* is the map induced by the inclusion map

$$L_n(X; p_X) \cong L_n(X \times \{p\}; p_X \times 1) \xrightarrow{i_*} L_n(X \times S^1; p_X \times 1).$$

The exactness follows easily from the exactness of the Mayer–Vietoris sequence. A splitting of B can be constructed by gluing two product cobordisms. □

Corollary 8.7 *Let T^n be the n–dimensional torus $S^1 \times \cdots \times S^1$. Then*

$$\begin{aligned} L_m(T^n; 1_{T^n}) &\cong \bigoplus_{r=0}^{n} \binom{n}{r} L_{m-r}(\mathbb{Z}) \\ &\cong L_m(\mathbb{Z}[\pi_1(T^n)]) \qquad (m \geq n). \end{aligned}$$

Proof Use Corollary 8.6 repeatedly to obtain

$$L_m(T^n; 1_{T^n}) \cong \bigoplus_{r=0}^{n} \binom{n}{r} L_{m-r}(\mathbb{Z}).$$

The isomorphism

$$\bigoplus_{r=0}^{n} \binom{n}{r} L_{m-r}(\mathbb{Z}) \cong L_m(\mathbb{Z}[\pi_1(T^n)])$$

is the well-known computation obtained geometrically by Shaneson and Wall, and algebraically by Novikov and Ranicki. □

9 Locally finite analogues

Up to this point, we considered only finitely generated modules and chain complexes. In this section we deal with infinitely generated objects; such objects arise naturally when we take the pullback of a finitely generated object via an infinite-sheeted covering map. We restrict ourselves to a very special case necessary for our application.

Definition 9.1 (Ranicki and Yamasaki [14, page 14]) Consider the product $M \times N$ of two spaces. A geometric module on $M \times N$ is said to be *M–finite* if, for any $y \in N$, there is a neighbourhood U of y in N such that $M \times U$ contains only finitely many basis elements; a projective module (A, p) on $M \times N$ is said to be *q–finite* if A is M–finite; a projective chain complex (C, p) on $M \times N$ is *M–finite* if each (C_r, p_r) is M–finite. (In [14], we used the terminology "*M–locally finite*", but this does not sound right and we decided to use "*M–finite*" instead.) When M is compact, M–finiteness is equivalent to the ordinary locally-finiteness.

Consider a control map $p_X\colon\ M \to X$ to a metric space X, and let N be another metric space. Give the maximum metric to the product $X \times N$, and let us use the map

$$p_X \times 1_N\colon\ M \times N \longrightarrow X \times N,$$

as the control map for $M \times N$.

Definition 9.2 For $\delta \ge \epsilon > 0$, $Y \subset X$, and a family $\mathcal{F}$ of subsets of X containing Y, define *M–finite (δ, ϵ)–controlled L–groups* $L_n^{M,\delta,\epsilon}(X \times N, Y \times N; p_X \times 1, R)$, and *$M$–finite (δ, ϵ)–controlled projective L–groups* $L_n^{M,\mathcal{F},\delta,\epsilon}(Y \times N; p_X \times 1, R)$ by requiring that every chain complexes concerned are M–finite.

All the materials up to Section 7 are valid for M–finite analogues. In the previous section, there are several places where we assumed X to be a finite polyhedron, and they may not automatically generalize to the M–finite case.

The most striking phenomenon about M–finite objects is the following vanishing result on the half line.

Proposition 9.3 *Let $p_X\colon M \to X$ be a control map, N a metric space, and give $N\times[0,\infty)$ the maximum metric. For any $\epsilon>0$ and $\delta\geq\epsilon$,*

$$L_n^{M,\delta,\epsilon}(X\times N\times[0,\infty);p_X\times 1,R)=0,$$
$$\widetilde{K}_0^{M,n,\epsilon}(X\times N\times[0,\infty);p_X\times 1,R)=0.$$

Proof This is done using repeated shifts towards infinity and the 'Eilenberg Swindle'. Let us consider the case of $L_n^{M,\delta,\epsilon}(X\times N\times[0,\infty);p_X\times 1,R)$. Let $J=[0,\infty)$ and define $T\colon M\times N\times J\to M\times N\times J$ by $T(x,x',t)=(x,x',t+\epsilon)$. Take an element $[c]\in L_n^{M,\delta,\epsilon}(X\times N\times J,p_X\times 1,R)$. It is zero, because there exist M–finite ϵ Poincaré cobordisms:

$$\begin{aligned} c \;&\sim\; c\oplus(T_\#(-c)\oplus T_\#^2(c))\oplus(T_\#^3(-c)\oplus T_\#^4(c))\oplus\cdots \\ &=\;(c\oplus T_\#(-c))\oplus(T_\#^2(c)\oplus T_\#^3(-c))\oplus\cdots\;\sim\;0. \end{aligned}$$

The proof for controlled $\widetilde{K}$ is similar. See the appendix to [14]. □

Thus, the analogue of Mayer–Vietoris type sequence (7.3) for the control map $p_X\times 1\colon M\times N\times\mathbb{R}\to X\times N\times\mathbb{R}$ with respect to the splitting $X\times N\times\mathbb{R}=X\times N\times(-\infty,0]\cup X\times N\times[0,\infty)$ reduces to

$$0\longrightarrow L_n^{M,\delta,\epsilon}(X\times N\times\mathbb{R};p_X\times 1,R)\xrightarrow{\partial}L_{n-1}^{M,p,\delta',\epsilon'}(X\times N\times I;p_X\times 1,R)\longrightarrow 0,$$

where $\delta'=\kappa_n\delta$, $\epsilon'=\kappa_n\epsilon$, $I=[-\kappa_n\delta,\kappa_n\delta]$, and the right hand side is the M–finite projective L–group $L_{n-1}^{M,\{\,\},\delta',\epsilon'}(X\times N\times I;p_X\times 1,R)$ corresponding to the empty family $\mathcal{F}=\{\,\}$.

A diagram chase shows that there exists a well-defined homomorphism

$$\beta\colon L_{n-1}^{M,p,\delta',\epsilon'}(X\times N\times I;p_X\times 1,R)\longrightarrow L_n^{M,\delta'',\epsilon''}(X\times N\times\mathbb{R};p_X\times 1,R),$$

where $\gamma''=\lambda_n\kappa_n\lambda_{n-1}\alpha\gamma'$ and $\epsilon''=\lambda_n\kappa_n\lambda_{n-1}\alpha\epsilon'$. The homomorphisms ∂ and β are stable inverses of each other; the compositions are both relax-control maps.

Note that, for any $\delta\geq\epsilon>0$, the retraction induces an isomorphism

$$L_{n-1}^{M,p,\delta,\epsilon}(X\times N\times I;p_X\times 1,R)\;\cong\;L_{n-1}^{M,p,\delta,\epsilon}(X\times N\times\{0\};p_X,R).$$

Thus, we have obtained:

Theorem 9.4 *Splitting along $X \times N \times \{0\}$ induces a stable isomorphism:*

$$\partial \colon\ L_n^{M,\delta,\epsilon}(X \times N \times \mathbb{R}; p_X \times 1, R) \longrightarrow L_{n-1}^{M,p,\delta',\epsilon'}(X \times N; p_X \times 1, R).$$

Now , as in the previous section, let us assume that X is a finite polyhedron and $p_X \colon\ M \to X$ is a fibration with a path-connected fiber F such that $\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = 0$ for every $k \geq 0$.

The following is an M–finite analogue of Proposition 8.1.

Proposition 9.5 *Let p_X be as above and $n \geq 0$, $k \geq 0$ be integers. Then there exist numbers $\epsilon_0 > 0$ and $0 < \mu \leq 1$ which depend on X, n, and k such that the relax-control maps*

$$\widetilde{K}_0^{M,n,\epsilon}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z}) \longrightarrow \widetilde{K}_0^{M,n,\epsilon'}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z})$$
$$\mathrm{Wh}^{M,n,\epsilon}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z}) \longrightarrow \mathrm{Wh}^{M,n,\epsilon'}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z})$$

is the zero map for any $\epsilon' \leq \epsilon_0$ and any $\epsilon \leq \mu\epsilon'$.

Proof First note that, since $X \times \mathbb{R}^k$ is not a finite polyhedron unless $k = 0$, the proof for Proposition 8.1 does not directly apply to the current situation.

Let us consider the Whitehead group case first. Since the $k = 0$ case was already treated in Proposition 8.1, let us suppose $k > 0$. Let T^k denote the k–torus $(S^1)^k$, and define $p_X^{(k)} \colon\ M \times T^k \to X$ to be the following composite map:

$$M \times T^k \xrightarrow{\text{projection}} M \xrightarrow{p_X} X.$$

By the Mayer–Vietoris type sequence for controlled K–theory, the group

$$\mathrm{Wh}^{M,n,\epsilon}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z})$$

is stably isomorphic to

$$\widetilde{K}_0^{M,n-1,\epsilon}(X \times \mathbb{R}^{k-1}; p_X \times 1, \mathbb{Z}),$$

which is also stably isomorphic to

$$\mathrm{Wh}^{M\times S^1,n,\epsilon}(X \times \mathbb{R}^{k-1}; p'_X \times 1, \mathbb{Z}).$$

The last statement is a locally-finite analogue of [14, 7.1]. The proof given there works equally well here. Therefore $\mathrm{Wh}^{M,n,\epsilon}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z})$ is stably isomorphic to

$$\mathrm{Wh}^{M\times T^k,n,\epsilon}(X; p_X^{(k)}, \mathbb{Z}),$$

for which the stable vanishing is already known. This completes the Whitehead group case.

The $\widetilde{K}_0$ case follows from the stable vanishing of

$$\mathrm{Wh}^{M,n+1,\epsilon}(X \times \mathbb{R}^{k+1}; p_X \times 1, \mathbb{Z}). \qquad \square$$

From this we get:

Proposition 9.6 *Assume that X is a finite polyhedron and $p_X\colon M \to X$ is a fibration with a path-connected fiber F such that $\mathrm{Wh}(\pi_1(F) \times \mathbb{Z}^k) = 0$ for every $k \geq 0$. Splitting along $X \times \mathbb{R}^{m-1} \times \{0\}$ induces a stable isomorphism*

$$\partial\colon L_n^{M,\delta,\epsilon}(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z}) \longrightarrow L_{n-1}^{M,\delta',\epsilon'}(X \times \mathbb{R}^{m-1}; p_X \times 1, \mathbb{Z}).$$

Corollary 9.7 *Let X and p_X be as above, then stability holds for $L_n^{M,\delta,\epsilon}(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z})$; that is, it is isomorphic to the limit*

$$L_n^M(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z}) = \lim_{0<\epsilon\ll\delta\to 0} L_n^{M,\delta,\epsilon}(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z})$$

when $0 < \epsilon \ll \delta$ and δ is sufficiently small.

Proof By the 4–periodicity, we may assume that $n > m$. Then the proposition above gives a stable isomorphism with $L_{n-m}^{\delta,\epsilon}(X; p_X, \mathbb{Z})$, and the result follows. $\square$

Corollary 9.8 *Let X and p_X be as above, then splitting along $X \times \mathbb{R}^{m-1} \times \{0\}$ induces an isomorphism*

$$\partial\colon L_n^M(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z}) \longrightarrow L_n^M(X \times \mathbb{R}^{m-1} \times \{0\}; p_X \times 1, \mathbb{Z}).$$

Proof Immediate from Proposition 9.6 and Corollary 9.7. $\square$

10 Controlled surgery obstructions

We discuss the controlled surgery obstructions and an application. We only consider the identity control maps on polyhedra or on the products of polyhedra and $\mathbb{R}^m$. X–finiteness on $X \times \mathbb{R}^m$ is the same as the usual local finiteness, so we use the following notation throughout this section:

$$\begin{aligned} L_n^{lf,\delta,\epsilon}(X \times \mathbb{R}^m) &= L_n^{X,\delta,\epsilon}(X \times \mathbb{R}^m; 1_X \times 1, \mathbb{Z}) \\ L_n^{lf}(X \times \mathbb{R}^m) &= L_n^X(X \times \mathbb{R}^m; 1_X \times 1, \mathbb{Z}) \end{aligned}$$

We omit the decoration '*lf*' when $m = 0$.

Let $(f,b)\colon M \to N$ be a degree 1 normal map between connected oriented closed *PL* manifolds of dimension n. Quadratic construction on this produces an element $\sigma_N(f,b) \in L_n^{\delta,\epsilon}(N)$ for arbitrarily small $\delta \gg \epsilon > 0$ (see Ranicki–Yamasaki [13]). By Proposition 8.2, this defines an element $\sigma_N(f,b) \in L_n^{lf}(N)$. This is the *controlled surgery obstruction* of (f,b), and its image via the forget-control map

$$L_n(N) \to L_n(\{pt.\}; N \to \{pt.\}, \mathbb{Z}) = L_n(\mathbb{Z}[\pi_1(N)])$$

is the ordinary surgery obstruction $\sigma(f,b)$ of (f,b). The controlled surgery obstruction $\sigma_N(f,b)$ vanishes, if (f,b) is normally bordant to a sufficiently small homotopy equivalence measured on N.

Similarly, if $(f,b)\colon V \to W$ is a degree 1 normal map between connected open oriented *PL* manifolds of dimension n, we obtain its *controlled surgery obstruction* $\sigma_W(f,b)$ in $L_n^{lf}(W)$.

Theorem 10.1 *Let X be a connected oriented closed PL manifold of dimension $4k$, and $f\colon V^n \to W^n = X \times \mathbb{R}^{n-4k}$ be a homeomorphism of open PL manifolds. Homotope f to produce a map $g\colon V \to W$ which is transverse regular to $X \times \{0\} \subset X \times \mathbb{R}^{n-4k}$. Then the PL submanifold $g^{-1}(X \times \{0\})$ of V and X have the same signature: $\sigma(g^{-1}(X \times \{0\})) = \sigma(X)$.*

Proof The homeomorphism f determines a degree 1 normal map F between V and W, and hence determines an element $\sigma_W(F) \in L_n^{lf}(X \times \mathbb{R}^{n-4k})$. Repeated application of splitting Corollary 9.8 induces an isomorphism

$$\partial^{n-4k}\colon L_n^{lf}(X \times \mathbb{R}^{n-4k}) \to L_{4k}(X).$$

The image of $\sigma_W(F)$ by this map is the controlled surgery obstruction $\sigma_X(g|,b)$ of the degree 1 normal *PL* map $(g|,b)\colon Y = g^{-1}(X \times \{0\}) \to X \times \{0\} = X$. Since f is a homeomorphism, F is normally bordant to arbitrarily small homotopy equivalences. Therefore, $\sigma_W(F)$ is zero and hence $\sigma_X(g|,b)$ is zero. This means that the ordinary surgery obstruction $\sigma(g|,b)$ is also zero. The equality $\sigma(Y) = \sigma(X)$ follows from this. □

Now we apply the above to prove the topological invariance of the rational Pontrjagin classes (see Novikov [3]).

Theorem 10.2 (S P Novikov) *If $h\colon M^n \to N^n$ is a homeomorphism between oriented closed PL manifolds, then $h^* p_i(N) = p_i(M)$, where p_i are the rational Pontrjagin classes.*

Proof Recall that the rational Pontrjagin classes $p_*(N) \in H^{4*}(N;\mathbb{Q})$ determine and are determined by the $\mathcal{L}$–genus $\mathcal{L}_*(N) \in H^{4*}(N;\mathbb{Q})$, and that the degree $4k$ component $\mathcal{L}_k(N) \in H^{4k}(N;\mathbb{Q})$ of the $\mathcal{L}$–genus is characterized by the property $\langle \mathcal{L}_k(N), x\rangle = \sigma(X)$ for $x \in \mathrm{im}([N, S^{n-4k}] \to H_{4k}(N;\mathbb{Q}))$, where $X^{4k} \subset N$ is the inverse image $f^{-1}(p)$ of some regular value $p \in S^{n-4k}$ of a map $f\colon\ N \to S^{n-4k}$ which represents the Poincaré dual of x and is PL transverse regular to p. Set $x' = (h^{-1})_*(x) \in H_{4k}(M;\mathbb{Q})$ and let us show that $\langle \mathcal{L}_k(M), x'\rangle = \langle \mathcal{L}_k(N), x\rangle$.

Since X is framed in N, it has an open PL neighborhood $W = X \times \mathbb{R}^{n-4k}$ in N. Let $V = h^{-1}(W) \subset M$, then h restricts to a homeomorphism $f\colon\ V \to W$. Homotope f to get a map g which is PL transverse regular to $X = X \times \{0\}$, and set Y to be the preimage $g^{-1}(X)$, then $\langle \mathcal{L}_k(M), x'\rangle = \sigma(Y)$ and this is equal to $\sigma(X) = \langle \mathcal{L}_k(N), x\rangle$ by Theorem 10.1. $\square$

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